LEONHARDI EULERI
OPERA OMNIA

SUB AUSPICIIS
SOCIETATIS SCIENTIARUM NATURALIUM
HELVETICAE

EDENDA CURAVERUNT

FERDINAND RUDIO
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SERIES PRIMA
OPERA MATHEMATICA
VOLUMEN UNDECIMUM

LIPSIAE ET BEROLINI
TYPIS ET IN AEDIBUS B. G. TEUBNERI
MCMXIII

LEONHARDI EULERI

INSTITUTIONES CALCULI INTEGRALIS

EDIDERUNT

FRIEDRICH ENGEL ET LUDWIG SCHLESINGER

VOLUMEN PRIMUM

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VORWORT DER HERAUSGEBER

Den Lehrgang der Analysis, mit dem EULER seine Zeitgenossen und Nachfahren beschenkt hat, eröffnet die im Jahre 1748 erschienene *Introductio in analysin infinitorum*, von der aber hier nur der erste Band in Betracht kommt, da sich der zweite mit der analytischen Geometrie beschäftigt. Es folgten 1755 die *Institutiones calculi differentialis* und in den Jahren 1768, 1769, 1770 die drei Bände der *Institutiones calculi integralis*.[1]) Diese Werke sind wiederholt neu aufgelegt und übersetzt worden, sie haben die ersten Schritte vieler Mathematiker geleitet, die später selbst Leuchten der Wissenschaft geworden sind, und ihre lehrende und bildende Kraft wirkt bis in die Gegenwart hinein noch immer weiter; ist ja noch 1885 eine neue deutsche Übersetzung der *Introductio* und 1895 eine neue (die vierte) Auflage des dritten Bandes des *Calculus integralis* erschienen. Auch heute gilt noch für jeden Jünger unserer Wissenschaft die Mahnung von LAPLACE: „Lisez EULER, lisez EULER, c'est notre maître à tous".[2])

Wir wissen aus einem Briefe an GOLDBACH vom 4. Juli 1744[3]), daß EULER schon im Alter von siebenundzwanzig Jahren begonnen hat, den *Calculus differentialis* zu schreiben, und in einem Brief an denselben Adressaten vom 6. August 1748[4]) heißt es: „... anjetzo wird von Mr. Bousquet[5]) meine Abhandlung vom Calculo differentiali gedruckt"; trotzdem ist das Werk erst sieben Jahre später erschienen.

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1) Eine Bibliographie dieser drei Werke sowie der späteren Auflagen und Übersetzungen findet sich am Schluß dieses Vorworts.

2) Berichtet von G. LIBRI in einer Anzeige der von FUSS herausgegebenen *Correspondance* (siehe die folgende Fußnote), Journal des Savants Janvier 1846, p. 51.

3) P. H. FUSS, *Correspondance mathématique et physique* **1**, St. Pétersbourg 1843, p. 279.

4) Ebenda p. 473.

5) Bei Bousquet, in Lausanne und Genf, waren auch die *Methodus inveniendi lineas curvas* (1744) und die *Introductio in analysin infinitorum* (1748) erschienen. Wie aus einem (noch nicht veröffentlichten) Briefe EULERS an G. CRAMER in Genf vom 2. November 1751 hervorgeht, ließ sich aber EULER das Manuskript des *Calculus differentialis* wieder zurückschicken. Er wurde dann bei Michaelis in Berlin „impensis academiae imperialis scientiarum Petropolitanae" 1755 gedruckt

Ähnlich ist es auch dem *Calculus integralis* ergangen. In einem aus Berlin vom 2. Oktober 1759 datierten Briefe an LAGRANGE schreibt EULER[1]): „Quoniam his gravissimis temporibus ab aliis negotiis vacavi, librum de calculo integrali conscribere coepi, quod opus iam pridem etiam meditatus, atque adeo Academiae Petropolitanae pollicitus, nunc igitur iam notabilem partem absolvi. Calculum integralem ita definivi, ut esset methodus functiones unius pluriumve variabilium inveniendi ex data differentialium vel primi vel altiorum graduum relatione; unde prout functiones sint vel unius, vel duarum pluriumve variabilium, totum opus in duos libros divisi, ubi quidem pro posteriori vix quicquam est cultum". Und in einem Briefe an GOLDBACH vom 17. Dezember 1763[2]) heißt es: „Schon vor einigen Monaten habe ich mein Werk von dem Calculo integrali, woran ich schon seit vielen Jahren gearbeitet, völlig zu Stande gebracht, und die Haudesche Buchhandlung allhier ist Willens dasselbe nächstens zu verlegen. Das Gerücht davon hatte einen jungen lehrbegierigen Menschen aus der Schweiz hierhergetrieben, welcher sich nichts anders als die Erlaubniß ausgebeten, dieses Werk abzuschreiben, und ist darauf wieder zurückgereiset. Das Wunderbarste dabey ist, daß dieser Mensch von seiner Profession ein Kürschner gewesen."

Trotzdem hat es noch fünf Jahre gedauert, bis der erste Band des *Calculus integralis* erscheinen konnte.

In einer Anzeige dieses ersten Bandes im Journal des Sçavans[3]) liest man: „Les élémens de calcul différentiel que M. EULER publia il y a quelques années, faisoient désirer depuis longtems le calcul intégral qui devoit en être la suite; et il y avoit plusieurs années que le Manuscrit étoit en état d'être publié, il y avoit eu même quelques feuilles d'imprimées; mais l'Auteur n'ayant pas été à même de veiller à l'Édition elle étoit trop incorrecte; M. EULER ayant été appellé à Pétersbourg, l'Académie a songé bientôt à procurer aux Géomètres cet important Ouvrage." Und eine ausführliche Besprechung in der Allgemeinen Deutschen Bibliothek[4]) beginnt mit den Sätzen: „Endlich haben wir also den ersten Theil des EULERSCHEN Integralcalculs vor uns, und zugleich die Hoffnung das ganze Werk bald vollständig zu erhalten. Es wird zwar in Petersburg gedruckt, allein das Manuscript war längst schon in Berlin fertig, und es hätte sowol der deutschen gelehrten Welt, als dem deutschen Buchhandel, zur Ehre gereicht, wenn es schon vor zehen und mehr Jahren in

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1) *Opera postuma* I, 1862, p. 558.

2) P. H. FUSS, *Correspondance* 1, p. 671. Die Jahreszahl 1763 und die Bemerkung über die gravissima tempora in dem Briefe an LAGRANGE zeigen, daß der *Calculus integralis* in gewissem Sinne als eine Frucht des Siebenjährigen Krieges zu betrachten ist.

3) Tome XLII, Décembre 1769, Nr. 14, p. 455—457 des Amsterdamer Nachdrucks, Novembre 1769 der Edition de Paris.

4) Des XI. Bandes zweytes Stück, Berlin und Stettin 1770, S. 6—16. Die Besprechung ist unterzeichnet E.

Deutschland einen Verleger gefunden hätte, oder wenigstens zu Petersburg nicht gewisse Hindernisse dazwischen gekommen wären. Indessen kam in Frankreich BOUGAINVILLES *Calcul intégral*[1]) heraus. Dieser wurde sogleich zu Wien latein nachgedruckt, und so wurden besonders die Klöster in Oberdeutschland und theils auch Italien damit versehen. Warum aber nicht mit dem EULERSCHEN Werke, mit welchem doch das BOUGAINVILLESCHE gar nicht zu vergleichen ist? .... Es muß .... daran liegen, daß die meisten bey den ersten Anfängen stehen bleiben. Ein anderer Grund mag seyn, daß man sich einbildete, Herr EULER könne sich von seiner erhabenen Sphäre nicht herab lassen. Allein es ist ganz umgekehrt. Hr. E. hat in dem Vortrag der ersten Anfänge eine ihm eigene und ganz vorzügliche Klarheit .... Freilich muß man nicht bey seinem Integralcalcul anfangen, sondern sich vorerst mit seiner Analysi finitorum und Differentialcalcul hinlänglich bekannt machen."

EULER war also noch im Besitze seiner Sehkraft, als er den *Calculus integralis* verfaßte, dagegen hat er von dem gedruckten Text keine Zeile mehr gesehen; ist er doch 1766, bald nach seiner Rückkehr nach Petersburg, völlig erblindet. Dies erklärt die nicht geringe Anzahl von Druckfehlern und Versehen anderer Art, die der Text des *Calculus integralis* aufweist und die auch in den späteren Auflagen nur zum Teil berichtigt sind.

In seinen drei analytischen Lehrbüchern hat EULER einen ansehnlichen Teil seiner eigenen Forschungen auf dem Gebiete der Analysis des Unendlichen mit den Ergebnissen seiner Vorgänger und Zeitgenossen zu einem System verarbeitet, das sich in seiner Gliederung und zum Teil auch in seinen Einzelausführungen bis auf den heutigen Tag als die klassische Gestaltung dieser Wissenschaft bewährt und erhalten hat.

Insbesondere hat es EULER vermocht, durch die Schöpfung einer bis ins Einzelne durchgebildeten und geschmeidigen Formelsprache die Unterschiede auszugleichen, die noch bis weit über die Mitte des achtzehnten Jahrhunderts hinaus zwischen den Methoden der festländischen und denen der englischen Schule bestanden haben. Er hat es verstanden, die eigenartigen Vorzüge der Differential- und der Fluxionsmethode miteinander zu verschmelzen, und so ist ihm, dem Germanen, im Verein mit seinem großen romanischen Zeitgenossen LAGRANGE die „Versöhnung" gelungen, die er im § 6 des *Calculus integralis* noch als „kaum zu erhoffen" bezeichnet hat.[2])

Die Art, wie EULER in der *Introductio* die rationalen Funktionen einerseits, die Exponentialfunktion und den Logarithmus, die trigonometrischen und die vom ihm zuerst ein-

1) LOUIS ANTOINE DE BOUGAINVILLE, le jeune, *Traité de calcul intégral pour servir de Suite à l'Analyse des Infiniment-Petits de M. le Marquis de L'HÔPITAL.* Paris 1754.

2) Haec diversitas loquendi ita iam usu invaluit, ut conciliatio vix unquam sit expectanda. S. 6 dieses Bandes.

geführten Kreisfunktionen andererseits behandelt, ist zum Vorbild für die ganze Funktionentheorie geworden, indem man bei jeder neuentdeckten Transzendenten sich bestrebt hat und sich noch bestrebt, das zu leisten, was EULER für die genannten elementaren Funktionen geleistet hat.

Die großzügige Einteilung der Integralrechnung, die EULER seinem Werke zugrunde legt, besteht auch heute noch zu Recht und man könnte ganz ungezwungen und naturgemäß alle Fortschritte, die diese Disziplin von EULER an bis auf unsere Tage gemacht hat, in die verschiedenen Abschnitte (Sectiones) und Kapitel des EULERSCHEN *Calculus integralis* einordnen.

Aus der Lehre von den gewöhnlichen Integralen (Quadraturen) werden (liber I, pars I, sectio I, cap. I—VI) diejenigen Integrale von elementaren Differentialen behandelt, die wieder durch elementare Funktionen ausdrückbar sind, und es werden Reihenentwickelungen vorgenommen, die entweder nach Potenzen von linearen Funktionen der Variabeln oder nach den Kosinus und Sinus der Vielfachen eines Winkels fortschreiten. Seine berühmten Untersuchungen über die Addition, Multiplikation und Teilung der trigonometrischen und elliptischen Integrale hat EULER mit aufgenommen; sie finden sich in dem Abschnitt über die gewöhnlichen Differentialgleichungen erster Ordnung (liber I, pars I, sectio II, cap. V, VI) und in dem *Supplementum* zum dritten Bande. Von höheren Transzendenten, die sich durch die Integration elementarer Funktionen ergeben, wird noch die Funktion $\int \frac{dz}{\log z}$, der von MASCHERONI so genannte Hyperlogarithmus, jetzt Integrallogarithmus genannt, betrachtet.

Von weittragender Bedeutung ist die Untersuchung der bestimmten Integrale (valores integralium, quos certis tantum casibus recipiunt, liber I, pars I, sectio I, cap. VIII, IX). Es werden zunächst Fälle angegeben, wo Integrale, die sich allgemein nicht elementar ausdrücken lassen, für besondere Werte der Veränderlichen durch bekannte Symbole darstellbar sind, und darauf bestimmte Integrale als Funktionen eines Parameters zur Definition neuer Transzendenten benutzt. Die letztere Art von Transzendenten erweist sich dann im zweiten Bande (liber I, pars II, sectio I. cap. X, vgl. auch besonders § 1016) als ein mächtiges Instrument für die Lösung von linearen Differentialgleichungen, insbesondere von solchen zweiter Ordnung, und man wird gerade diese Untersuchungen zu den bedeutsamsten Förderungen zählen dürfen, die die Integralrechnung und die Funktionentheorie EULER verdanken. Daß diese Integration per quadraturas curvarum der mittels unendlicher Reihen (liber I, pars II, sectio I, cap. VII, VIII, XI) durch ihren meist größeren Geltungsbereich überlegen ist, war EULER wohl bewußt; er hat auch großen Wert auf diese seine Leistung gelegt, sagt er doch (§ 1016) darüber: Hoc autem argumentum fere prorsus est novum, neque a quoquam adhuc pertractatum, si quidem nonnulla specimina, quae equidem iam dudum dedi, excipiantur; ex quo dubitare non licet, quin ista methodus, si diligentius excolatur, aliquando forte praeclara incrementa in Analysin sit allatura“.

Daß für Differentialgleichungen erster Ordnung die Trennung der Veränderlichen und allgemeiner die Zurückführung der Integration auf gewöhnliche Integrale (Quadraturen) selbst in den Fällen, wo sie möglich ist, nicht immer die endgültige Lösung der durch die Differentialgleichung gestellten Aufgabe liefert, zeigt nichts deutlicher als das Beispiel der nach EULER benannten Differentialgleichung, die auf das Additionstheorem der elliptischen Integrale führt. In der Tat ist ja bei dieser die Trennung der Veränderlichen schon vollzogen und die Integration durch Quadraturen unmittelbar gegeben, aber ihr wahrer Inhalt tritt erst in der von EULER geleisteten Integration durch eine algebraische Gleichung zu Tage. Das Verfahren, das EULER zur Herstellung dieses algebraischen Integrals einschlägt, zeigt deutlich, daß er sich schon von dem Grundsatze hat leiten lassen, den zuerst ABEL in Worte gekleidet hat, nämlich die Probleme so zu fassen, daß sie lösbar werden.

Nach demselben Grundsatze verfährt EULER zum Teil auch in den Kapiteln, die über die Integration der Differentialgleichungen durch Multiplikatoren handeln, eine Methode, auf die er sehr großes Gewicht legt und die er sowohl auf Gleichungen erster als auf solche höherer Ordnung anwendet (liber I, pars I, sectio II, cap. II, III und liber I, pars II, sectio I, cap. V, VI). Wir denken dabei an seine Untersuchungen über Differentialgleichungen von vorgeschriebener Form, die durch Multiplikatoren von vorgeschriebener Form integrabel werden sollen. Obwohl er mit Hilfe von Multiplikatoren eine Anzahl einfacher Typen von Differentialgleichungen integriert hat, so kann diese Methode der integrierenden Faktoren doch nicht als ein Hilfsmittel von erheblicher Tragweite für die wirkliche Integration gelten, sondern sie hat mehr die Bedeutung eines ordnenden Prinzips.

In der Lehre von den partikulären Integralen (liber I, pars I, sectio II, cap. IV) hat EULER mehr das Verdienst, auf die zu lösenden Aufgaben hingewiesen, als sie wirklich bezwungen zu haben. Das Wesen der Lösungen, die man gemeinhin als singuläre bezeichnet, hat er noch nicht erkannt.

Bei der Anwendung unendlicher Reihen (Potenzreihen) zur Integration von Differentialgleichungen erster Ordnung (liber I, pars I, sectio II, cap. VII, § 655 ff.) und linearer Differentialgleichungen zweiter Ordnung (liber I, pars II, sectio I, cap. VII, VIII, XI) begründet EULER das Verfahren der unbestimmten Koeffizienten, indem er den Inhalt der Differentialgleichung in eine Rekursionsformel für die Reihenkoeffizienten umsetzt. Er vollzieht damit den ersten wichtigen Schritt, der die Entwickelung der analytischen Theorie der gewöhnlichen Differentialgleichungen einleitet. Es ist auch bemerkenswert, daß die von EULER betrachteten Reihen stets für hinreichend kleine Werte des Inkrements konvergieren; immer divergente Potenzreihen, die gewöhnlichen Differentialgleichungen genügen können, treten, wie es scheint, bei EULER überhaupt nicht auf.

Die Integration mittels unendlicher Reihen und in noch höherem Maße die Kapitel des *Calculus integralis*, in denen EULER zuerst für die gewöhnliche Quadratur (liber I, pars I, sectio I, cap. VII, § 650—657), dann für Differentialgleichungen erster Ordnung (liber I, pars I, sectio II, cap. VII) und für Differentialgleichungen zweiter Ordnung (liber I, pars II, sectio I, cap. XII) die Integration mittels Annäherung (per approximationem) behandelt, gibt Veranlassung, den Standpunkt zu beleuchten, den EULER zu den quantitativen Methoden eingenommen hat, die einer im Sinne der antiken Geometer strengen Behandlung der Infinitesimalrechnung zugrunde liegen, einer Behandlung, wie sie uns erst das neunzehnte Jahrhundert gebracht hat.

Daß bei EULER scharfe Eigenschaftsbestimmungen für die von ihm betrachteten Funktionen (Stetigkeit, Differentiierbarkeit, analytischer Charakter) fehlen, ist nicht zu verwundern; damit hängt es auch zusammen, daß er Existenzfragen niemals durch a priori zu gebende Beweise zu erledigen sucht. Für die Art wie EULER solchen Fragen begegnet, ist charakteristisch, was er (*Introductio*, t. I cap. IV, § 59) auf die Frage erwidert, ob sich wohl jede Funktion in eine unendliche Potenzreihe entwickeln lasse: „si quis dubitet", sagt er, „hoc dubium per ipsam evolutionem cuiusque functionis tolletur". Von diesem Standpunkte aus würde die Existenz des Integrals einer elementaren Funktion, die Existenz der Lösung einer Differentialgleichung, deren Koeffizienten elementare Funktionen sind, in der Weise zu beweisen sein, daß man die Integration ausführt, sei es durch elementare Funktionen oder, wenn das nicht geht, durch unendliche Reihen oder endlich noch allgemeiner „per approximationem", was soviel heißt wie durch das Interpolationsverfahren, also nach der Methode, die erst von CAUCHY und LIPSCHITZ wirklich durchgeführt worden ist. Natürlich ist dagegen nichts zu erinnern, da sich dieser Standpunkt von dem unsrigen eigentlich nur in der Formulierung unterscheidet; es ist eben vorkantisch, nicht a priori sondern a posteriori zu prüfen und zu erkennen. In der Tat finden wir auch bei EULER nicht nur die formalen Ansätze für die Integration durch Potenzreihen, sondern man kann sogar in den § 317—322 und 656—660, wo die NEWTONSCHE Näherungsmethode auf Differentialgleichungen übertragen wird, die Keime der analytischen Fortsetzung erkennen. Wir finden ferner in den bereits erwähnten Kapiteln über die Integration durch Annäherung den formalen Algorithmus für die Summendefinition des Integrals und für das CAUCHY-LIPSCHITZSCHE Interpolationsverfahren bei Differentialgleichungen; was man aber bei EULER vermißt, ist die quantitative Abschätzung dessen, wie weit man jene formalen Prozesse fortsetzen muß, um einen vorgeschriebenen Genauigkeitsgrad zu erzielen, das heißt also die Anwendung der strengen Methoden der Griechen.

Hätte EULER gesucht, den Grad der Annäherung, die Größe des begangenen Fehlers, wovon er immer wieder spricht (siehe die angeführten Stellen), wirklich zu bestimmen, so hätten sich ihm die Grenzen ergeben, innerhalb derer seine Integration durch Reihen oder durch Annäherung möglich ist, und er würde statt bloß formaler Ansätze wirkliche quantitative Integrationen geleistet haben.

Es ist eine Frage, die die Geschichte der Mathematik noch zu klären hat, durch welche Ursachen die strengen Methoden der Griechen im Laufe der Zeit außer Gebrauch gekommen sind, und wodurch das Gefühl für die Unentbehrlichkeit dieser Methoden später wieder geweckt worden ist. Bei EULER und LAGRANGE fehlen diese Methoden noch vollständig, wenige Jahrzehnte später sind sie bei GAUSS, BOLZANO, CAUCHY, ABEL mit vollem Bewußtsein wieder in Gebrauch. Wir müssen uns hier mit der Feststellung begnügen, daß EULER als Kind seiner Zeit sich jener Methoden nicht bedient hat, daß er darum nicht imstande war, sich über die Konvergenz unendlicher Prozesse deutliche Vorstellungen zu bilden[1]) und daß ihm aus demselben Grunde gewisse fundamentale Sätze keines Beweises bedürftig erschienen sind, wie z. B. der Satz (*Introductio*, t. I cap. II, § 33), daß eine ganze rationale Funktion von x, die für $x=a$ den Wert A, für $x=b$ den Wert B annimmt, von A zu B nicht übergehen kann, „nisi per omnes valores medios transeundo".

Daß gerade der *Calculus integralis* EULERS trotz der veränderten Auffassung, die sich seit seinem Erscheinen in bezug auf die Bewertung der quantitativen Abschätzungsmethoden herausgebildet hat, seine Bedeutung als Lehrbuch und Handbuch zu behaupten vermag, liegt daran, daß sich für die darin behandelten besonderen Probleme die fehlenden Konvergenzbetrachtungen leicht ergänzen lassen; die auf die Approximationen bezüglichen Kapitel würden allerdings einer tiefergreifenden Umarbeitung bedürfen und können in ihrer gegenwärtigen Gestalt nur geschichtliches Interesse beanspruchen.

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1) So heißt es z. B. am Anfang der Abhandlung 247 (des ENESTRÖMSCHEN Verzeichnisses): *De seriebus divergentibus*, Novi comment. acad. sc. Petrop. 5 1754/55, p. 205 (*LEONHARDI EULERI Opera omnia*, series I, vol. 14): Cum series convergentes ita definiantur, ut constent terminis continuo decrescentibus, qui tandem, si series in infinitum processerit, penitus evanescant... und aus dem § 9 dieser Abhandlung (a. a. O. p. 210) geht hervor, daß EULER annimmt, eine so beschaffene Reihe besäße eine Summe in demselben Sinne, wie etwa $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\text{etc.}=2$ ist, d. h. „quod quo plures istius seriei terminos actu addamus, eo propius nos ad binarium pervenire". Es ist von Interesse, demgegenüber darauf hinzuweisen, daß die Auffassung, der EULER in dieser selben Abhandlung in bezug auf die *divergenten* Reihen Ausdruck gibt (z. B. § 11, p. 212 a. a. O.) derjenigen sehr nahesteht, die BOREL in seinem Werke *Leçons sur les séries divergentes*, Paris 1901, (siehe besonders chap. IV) vertritt. Wenn nämlich EULER daselbst von der *expressio finita* spricht, *ex qua series proposita resultet*, so wird man darin das Tasten nach dem Begriffe erkennen, den wir heute als die monogene Funktion einer komplexen Veränderlichen bezeichnen, die etwa aus einer in beschränktem Bereiche konvergenten Potenzreihe durch analytische Fortsetzung entspringt. „Si igitur" heißt es dann „receptam summae notionem ita tantum immutemus, ut dicamus, cuiusque seriei summam esse expressionem finitam, ex cuius evolutione illa ipsa series nascatur, omnes difficultates, quae ab utraque parte [nämlich Gegnern und Anhängern des Rechnens mit divergenten Reihen] sunt commotae, sponte evanescent."

Der dritte Band des *Calculus integralis* ist den partiellen Differentialgleichungen erster und höherer Ordnung gewidmet, und auch hier ist man überrascht über die Fülle von verschiedenartigen Gleichungen, deren Integration gelingt. Diese Integrationen beruhen in der Regel darauf, daß ein Differentialausdruck von der Form VdU nur integrabel sein kann, wenn V eine Funktion von U allein ist, und EULER ist unerschöpflich in immer neuen Anwendungen dieses Grundgedankens. Aber die allgemeine Zurückführung der partiellen Differentialgleichungen erster Ordnung auf gewöhnliche konnte ihm nicht gelingen, weil er sie noch nicht einmal für die allgemeine lineare partielle Differentialgleichung erster Ordnung zu leisten vermochte.

Außer dem schon vorhin erwähnten *Supplementum* enthält der dritte Band noch eine *Appendix* über den *Calculus variationum.* EULER wird hierdurch der Tatsache gerecht, daß die von ihm auf geometrischem Wege entwickelte Lehre von den größten und kleinsten Werten der Integrale durch den von LAGRANGE geschaffenen Kalkül zu einem Teile der Differential- und Integralrechnung geworden war.

Der vorliegenden Ausgabe des *Calculus integralis* liegt die Urausgabe zugrunde, die einzige, die bei Lebzeiten EULERS erschienen ist. Und zwar beschränkt sich unsere Ausgabe auf die ursprünglich veröffentlichten drei Bände; der vierte Band, der unter dem Titel „*Institutionum calculi integralis volumen quartum, continens supplementa partim inedita partim iam in operibus academiae imperialis scientiarum Petropolitanae impressa*“ im Jahre 1794, also nach EULERS Tode herausgegeben worden ist, enthält achtundzwanzig Abhandlungen, die auf andere Bände der Gesamtausgabe von EULERS Werken verteilt worden sind[1]). Die Urausgabe wurde aber mit den späteren Auflagen und mit der deutschen Übersetzung von SALOMON verglichen; die große Sorgfalt, mit der die Herren Redaktoren die Herausgeber bei der Korrektur unterstützt haben, läßt hoffen, daß es gelungen sei, das alte Meisterwerk EULERS in durchaus korrekter Form neu erstehen zu lassen.

Offenbare Druckfehler wurden stillschweigend ausgemerzt, bei kleineren Rechenfehlern der Text verbessert und der Urtext in einer Fußnote angegeben. Einschaltungen der Herausgeber sind in eckige Klammern [] eingeschlossen. Einige Fußnoten geben teils Literaturangaben, teils auch Hinweise auf Versehen, die im Texte vorkommen; von erläuternden und ergänzenden Anmerkungen mußte im Sinne der für diese Gesamtausgabe geltenden Grundsätze Abstand genommen worden. Dagegen ist es möglich geworden, die beiden Teile der

1) Dieser vierte Band ist in allen spätern Auflagen mit abgedruckt und auch in die Übersetzung von SALOMON mit aufgenommen worden. Wir haben ihn aber aus der unten folgenden Bibliographie weggelassen.

Adnotationes ad calculum integralem Euleri von LORENZO MASCHERONI, die 1790 und 1792 in Pavia erschienen und zur Zeit recht selten geworden sind, unserer Ausgabe einzuverleiben. Sie folgen im zwölften Bande der ersten Serie auf den zweiten Band des *Calculus integralis*, der den *Liber prior* des ganzen Werkes abschließt, da sie sich ausschließlich auf diesen beziehen.

Es möge auch an dieser Stelle Sr. Exzellenz dem Königlich italienischen Minister des öffentlichen Unterrichts der ehrerbietigste Dank dafür ausgesprochen werden, daß er für die Aufnahme der MASCHERONISCHEN *Adnotationes* in die Eulerausgabe einen Staatszuschuß gewährt hat; nicht minder danken wir der *Società italiana per il progresso delle Scienze* und den Herren GINO LORIA und VITO VOLTERRA für ihr wirkungsvolles Eintreten in dieser Angelegenheit.

Was die Verteilung der Arbeit auf die beiden Herausgeber anlangt, so hat vom ersten Bande SCHLESINGER den ersten, ENGEL den zweiten und dritten Abschnitt bearbeitet. Den zweiten Band und die *Adnotationes* von MASCHERONI wird SCHLESINGER bearbeiten, den dritten Band ENGEL mit Ausnahme des *Supplementum.*

Gießen, den 26. Juni 1913.

L. SCHLESINGER. F. ENGEL.

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INSTITUTIONUM
CALCULI INTEGRALIS

VOLUMEN PRIMUM

INSTITVTIONVM CALCVLI INTEGRALIS

VOLVMEN PRIMVM

IN QVO METHODVS INTEGRANDI A PRIMIS PRINCIPIIS VSQVE AD INTEGRATIONEM AEQVATIONVM DIFFERENTIALIVM PRIMI GRADVS PERTRACTATVR.

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PETROPOLI

Impenſis Academiae Imperialis Scientiarum

1768.

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PRAENOTANDA

DE CALCULO INTEGRALI IN GENERE

DEFINITIO 1

1. *Calculus integralis est methodus ex data differentialium relatione inveniendi relationem ipsarum quantitatum; et operatio, qua hoc praestatur, integratio vocari solet.*

COROLLARIUM 1

2. Cum igitur calculus differentialis ex data relatione quantitatum variabilium relationem differentialium investigare doceat, calculus integralis methodum inversam suppeditat.

COROLLARIUM 2

3. Quemadmodum scilicet in Analysi perpetuo binae operationes sibi opponuntur, veluti subtractio additioni, divisio multiplicationi, extractio radicum evectioni ad potestates, ita etiam simili ratione calculus integralis calculo differentiali opponitur.

COROLLARIUM 3

4. Proposita relatione quacunque inter binas quantitates variabiles x et y in calculo differentiali methodus traditur rationem differentialium $dy:dx$ investigandi; sin autem vicissim ex hac differentialium ratione ipsa quantitatum x et y relatio sit definienda, hoc opus calculo integrali tribuitur.

SCHOLION 1

5. In calculo differentiali iam notavi quaestionem de differentialibus non absolute sed relative esse intelligendam, ita ut, si y fuerit functio quaecunque ipsius x, non tam ipsum eius differentiale dy quam eius ratio ad differentiale dx sit definienda. Cum enim omnia differentialia per se sint nihilo aequalia, quaecunque functio y fuerit ipsius x, semper est $dy = 0$ neque sic quicquam amplius absolute quaeri posset. Verum quaestio ita rite proponi debet, ut, dum x incrementum capit infinite parvum adeoque evanescens dx, definiatur ratio incrementi functionis y, quod inde capiet, ad istud dx; etsi enim utrumque est $= 0$, tamen ratio certa inter ea intercedit, quae in calculo differentiali proprie investigatur. Ita si fuerit $y = xx$, in calculo differentiali ostenditur esse $\frac{dy}{dx} = 2x$ neque hanc incrementorum rationem esse veram, nisi incrementum dx, ex quo dy nascitur, nihilo aequale statuatur. Verum tamen hac vera differentialium notione observata locutiones communes, quibus differentialia quasi absolute enunciantur, tolerari possunt, dummodo semper in mente saltem ad veritatem referantur. Recte ergo dicimus, si $y = xx$, fore $dy = 2xdx$, tametsi falsum non esset, si quis diceret $dy = 3xdx$ vel $dy = 4xdx$, quoniam ob $dx = 0$ et $dy = 0$ hae aequalitates aeque subsisterent; sed prima sola rationi verae $\frac{dy}{dx} = 2x$ est consentanea.

SCHOLION 2

6. Quemadmodum calculus differentialis apud Anglos methodus fluxionum appellatur, ita calculus integralis ab iis methodus fluxionum inversa vocari solet, quandoquidem a fluxionibus ad quantitates fluentes revertitur. Quas enim nos quantitates variabiles vocamus, eas Angli nomine magis idoneo quantitates fluentes vocant et earum incrementa infinite parva seu evanescentia fluxiones nominant, ita ut fluxiones ipsis idem sint, quod nobis differentialia. Haec diversitas loquendi ita iam usu invaluit, ut conciliatio vix unquam sit expectanda; equidem Anglos in formulis loquendi lubenter imitarer, sed signa, quibus nos utimur, illorum signis longe anteferenda videntur. Verum cum tot iam libri utraque ratione conscripti prodierint, huiusmodi conciliatio nullum usum esset habitura.

DEFINITIO 2

7. *Cum functionis cuiuscunque ipsius x differentiale huiusmodi habeat formam Xdx, proposita tali forma differentiali Xdx, in qua X sit functio quaecunque ipsius x, illa functio, cuius differentiale est $= Xdx$, huius vocatur integrale et praefixo signo $\int$ indicari solet, ita ut $\int Xdx$ eam denotet quantitatem variabilem, cuius differentiale est $= Xdx$.*

COROLLARIUM 1

8. Quemadmodum ergo propositae formulae differentialis Xdx integrale seu ea functio ipsius x, cuius differentiale est $= Xdx$, quae hac scriptura $\int Xdx$ indicatur, investigari debeat, in calculo integrali est explicandum.

COROLLARIUM 2

9. Uti ergo littera d signum est differentiationis, ita littera $\int$ pro signo integrationis utimur sicque haec duo signa sibi mutuo opponuntur et quasi se destruunt, scilicet $\int dX$ erit $= X$, quia ea quantitas denotatur, cuius differentiale est dX, quae utique est X.

COROLLARIUM 3

10. Cum igitur harum ipsius x functionum x^2, x^n, $\sqrt{(aa-xx)}$ differentialia sint $2xdx$, $nx^{n-1}dx$, $\frac{-xdx}{\sqrt{(aa-xx)}}$, signo integrationis $\int$ adhibendo patet fore

$$\int 2xdx = xx, \quad \int nx^{n-1}dx = x^n, \quad \int \frac{-xdx}{\sqrt{(aa-xx)}} = \sqrt{(aa-xx)},$$

unde usus huius signi clarius perspicitur.

SCHOLION 1

11. Hic unica tantum quantitas variabilis in computum ingredi videtur, cum tamen statuamus tam in calculo differentiali quam integrali semper rationem duorum pluriumve differentialium spectari. Verum etsi hic una tantum quantitas variabilis x apparet, tamen revera duae considerantur; altera enim est ipsa illa functio, cuius differentiale sumimus esse Xdx; quae si

designetur littera y, erit $dy = Xdx$ seu $\frac{dy}{dx} = X$, ita ut hic omnino ratio differentialium $dy : dx$ proponatur, quae est $= X$, indeque erit $y = \int Xdx$; hoc autem integrale non tam ex ipso differentiali Xdx, quod utique est $= 0$, quam ex eius ratione ad dx inveniri est censendum. Caeterum hoc signum $\int$ vocabulo *summae* efferri solet, quod ex conceptu parum idoneo, quo integrale tanquam summa omnium differentialium spectatur, est natum; neque maiore iure admitti potest, quam vulgo lineae ex punctis constare concipi solent.

SCHOLION 2

12. At calculus integralis multo latius quam ad huiusmodi formulas integrandas patet, quae unicam variabilem complectuntur. Quemadmodum enim hic functio unius variabilis x ex data differentialis forma investigatur, ita calculus integralis quoque extendi debet ad functiones duarum pluriumve variabilium investigandas, cum relatio quaedam differentialium fuerit proposita. Deinde calculus integralis non solum ad differentialia primi ordinis adstringitur, sed etiam praecepta tradere debet, quorum ope functiones tam unius quam duarum pluriumve variabilium investigari queant, cum relatio quaedam differentialium secundi altiorisve cuiusdam ordinis fuerit data. Atque hanc ob rem definitionem calculi integralis ita instruximus, ut omnes huiusmodi investigationes in se complecteretur; differentialia enim cuiusque ordinis intelligi debent et voce *relationis*, quae inter ea proponatur, sum usus, ut latius pateret voce *rationis*, quae tantum duorum differentialium comparationem indicare videatur. Ex his ergo divisionem calculi integralis constituere poterimus.

DEFINITIO 3

13. *Calculus integralis dividitur in duas partes, quarum prior tradit methodum functionem unius variabilis inveniendi ex data quadam relatione inter eius differentialia tam primi quam altiorum ordinum.*

Pars autem altera methodum continet functionem duarum pluriumve variabilium inveniendi, cum relatio inter eius differentialia sive primi sive altioris cuiusdam gradus fuerit proposita.

COROLLARIUM 1

14. Prout ergo functio ex data differentialium relatione invenienda vel unicam variabilem complectitur vel duas pluresve, inde calculus integralis commode in duas partes principales dispescitur, quibus exponendis duos libros destinamus.

COROLLARIUM 2

15. Semper igitur calculus integralis in inventione functionum vel unius vel plurium variabilium versatur, cum scilicet relatio quaepiam inter eius differentialia sive altioris cuiuspiam ordinis fuerit proposita.

SCHOLION

16. Cum hic primam partem calculi integralis in investigatione functionum unicae variabilis ex data differentialium relatione constituamus, plures partes pro numero variabilium functionem ingredientium constitui debere videantur, ita ut pars secunda functiones duarum variabilium, tertia trium, quarta quatuor etc. complectatur. Verum pro his posterioribus partibus methodus fere eadem requiritur, ita ut, si inventio functionum duas variabiles involventium fuerit in potestate, via ad eas, quae plures variabiles implicant, satis sit patefacta; unde inventionem eiusmodi functionum, quae duas pluresve variabiles continent, commode coniungimus indeque unicam partem calculi integralis constituimus posteriori libro tractandam.

Caeterum haec altera pars in elementis adhuc nusquam est tractata, etiamsi eius usus in Mechanica ac praecipue in doctrina fluidorum maximi sit usus. Quocirca cum in hoc genere praeter prima rudimenta vix quicquam sit exploratum, noster secundus liber de calculo integrali admodum erit sterilis ac praeter commemorationem eorum, quae adhuc desiderantur, parum erit expectandum; verum hoc ipsum ad scientiae incrementum multum conferre videtur.

DEFINITIO 4

17. *Uterque de calculo integrali liber commode subdividitur in partes pro gradu differentialium, ex quorum relatione functionem quaesitam investigari oportet. Ita prima pars versatur in relatione differentialium primi gradus, secunda in relatione differentialium secundi gradus, quorsum etiam differentialia altiorum graduum ob tenuitatem eorum, quae adhuc sunt investigata, referri possunt.*

COROLLARIUM 1

18. Uterque ergo liber constabit duabus partibus, in quarum priore relatio inter differentialia primi gradus proposita considerabitur, in posteriore vero eiusmodi integrationes occurrent, ubi relatio inter differentialia secundi altiorumve graduum proponitur.

COROLLARIUM 2

19. In primi ergo libri parte prima eiusmodi functio variabilis x invenienda proponitur, ut posita ea functione $= y$ et $\frac{dy}{dx} = p$ relatio quaecunque data inter has tres quantitates x, y et p adimpleatur, seu proposita quacunque aequatione inter has ternas quantitates ut indoles functionis y seu aequatio inter x et y tantum, exclusa p, eruatur.

COROLLARIUM 3

20. Posterioris autem partis primi libri quaestiones ita erunt comparatae, ut posito $\frac{dy}{dx} = p$, $\frac{dp}{dx} = q$, $\frac{dq}{dx} = r$ etc., si proponatur aequatio quaecunque inter quantitates x, y, p, q, r etc., indoles functionis y per x seu aequatio inter x et y eliciatur.

SCHOLION 1

21. Quae adhuc in calculo integrali sunt elaborata, maximam partem ad libri primi partem primam sunt referenda, in qua excolenda Geometrae imprimis operam suam collocarunt; pauca sunt, quae in parte posteriore sunt praestita, et alter liber, quem secundum fecimus, etiamnunc fere vacuus est relictus. Prima autem pars libri primi, in qua potissimum nostra tractatio consumetur, denuo in plures sectiones distinguitur pro modo relationis, quae inter quantitates x, y et $p = \frac{dy}{dx}$ proponitur. Relatio enim prae caeteris simplicissima est, quando $p = \frac{dy}{dx}$ aequatur functioni cuipiam ipsius x; qua posita $= X$, ut sit $\frac{dy}{dx} = X$ seu $dy = Xdx$, totum negotium in integratione formulae differentialis Xdx absolvitur; huius operationis iam supra mentionem fecimus, quae vulgo sub titulo integrationis formularum differentialium simplicium seu unicam variabilem involventium tractari solet. Eodem res rediret, si $p = \frac{dy}{dx}$ aequaretur functioni ipsius y tantum, quandoquidem quantitates x et y ita inter se reciprocantur, ut altera tanquam functio alterius spectari possit;

haec ergo ad sectionem primam referentur. Sin autem $p = \frac{dy}{dx}$ aequetur expressioni ambas quantitates x et y involventi, aequatio habetur differentialis huius formae $Pdx + Qdy = 0$, ubi P et Q sunt expressiones quaecunque ex x, y et constantibus conflatae. Quanquam autem Geometrae multum in huiusmodi aequationum integratione desudarunt, tamen vix ultra quosdam casus satis particulares sunt progressi. Sin autem p magis complicate per x et y determinatur, ut eius valor explicite exhiberi nequeat, veluti si fuerit

$$p^5 = xxp^3 - xyp + x^5 - y^5,$$

ne via quidem constat tentanda, quomodo inde relatio inter x et y investigari queat; pauca ergo, quae hic tradere licebit, cum praecedentibus secundam sectionem primae partis libri primi occupabunt. Ita ex universa nostra tractatione magis patebit, quid adhuc in calculo integrali desideretur, quam quid iam sit expeditum, cum hoc prae illo ut minima quaedam particula sit spectandum.

SCHOLION 2

22. In singulis partibus, quas enarravimus, fieri etiam solet, ut non solum una quaedam functio, sed etiam simul plures investigentur, ita ut neutra sine reliquis definiri possit, quemadmodum in Algebra communi usu venit, ut ad solutionem problematis plures incognitae in calculum sint introducendae, quae deinceps per totidem aequationes determinentur. Veluti si eiusmodi binae functiones y et z ipsius x sint inveniendae, ut sit

$$xdy + azzdx = 0 \quad \text{et} \quad xxdz + bxydy = cdy,$$

hinc novae subdivisiones nostrae tractationis constitui possent. Verum quia hic ut in Algebra communi totum negotium ad eliminationem unius litterae revocatur, ut deinceps duae tantum variabiles in una aequatione supersint, hinc tractatio non multiplicanda videtur.

SCHOLION 3

23. In secundo libro calculi integralis, quo functio duarum pluriumve variabilium ex data differentialium relatione investigatur, multo maior quaestionum varietas locum habet. Sit enim z functio binarum variabilium x et t investiganda, et cum $\left(\frac{dz}{dx}\right)$ denotet rationem eius differentialis ad dx, si sola x pro variabili habeatur, at $\left(\frac{dz}{dt}\right)$ rationem eius differentialis ad dt, si sola t

variabilis sumatur, prima pars eiusmodi continebit quaestiones, in quibus certa quaedam relatio inter quantitates x, t, z et $\left(\frac{dz}{dx}\right)$, $\left(\frac{dz}{dt}\right)$ proponitur, et quaestio huc redit, ut hinc aequatio inter solas quantitates x, t et z eruatur; inde enim qualis z sit functio ipsarum x et t patebit. In secunda parte praeter has formulas $\left(\frac{dz}{dx}\right)$ et $\left(\frac{dz}{dt}\right)$ etiam istae $\left(\frac{ddz}{dxdx}\right)$, $\left(\frac{ddz}{dxdt}\right)$ et $\left(\frac{ddz}{dtdt}\right)$ in computum ingredientur, quarum significatio ita est intelligenda, ut positis prioribus $\left(\frac{dz}{dx}\right)=p$ et $\left(\frac{dz}{dt}\right)=q$, ubi p et q iterum certae erunt functiones ipsorum x et t, futurum sit simili expressionis modo

$$\left(\frac{ddz}{dxdx}\right)=\left(\frac{dp}{dx}\right),\quad \left(\frac{ddz}{dxdt}\right)=\left(\frac{dp}{dt}\right)=\left(\frac{dq}{dx}\right),\quad \left(\frac{ddz}{dtdt}\right)=\left(\frac{dq}{dt}\right).$$

Proposita ergo relatione inter has formulas et praecedentes simulque ipsas quantitates x, t et z, aequatio inter ternas istas quantitates solas x, t et z erui debet. Huiusmodi quaestiones frequenter occurrunt in Mechanica et Hydraulica, quando motus corporum flexibilium et fluidorum indagatur; ex quo maxime est optandum, ut haec altera sectio secundi libri calculi integralis omni cura excolatur. Neque vero opus erit, ut hanc investigationem ad differentialia altiora extendamus, cum nullae adhuc quaestiones sint tractatae, quae tanta calculi incrementa desiderent.

DEFINITIO 5

24. *Si functiones, quae in calculo integrali ex relatione differentialium quaeruntur, algebraice exhiberi nequeant, tum eae vocantur transcendentes, quandoquidem earum ratio vires Analyseos communis transcendit.*

COROLLARIUM 1

25. Quoties ergo integratio non succedit, toties functio, quae per integrationem quaeritur, pro transcendente est habenda. Ita si formula differentialis Xdx integrationem non admittit, eius integrale, quod ita indicari solet $\int Xdx$, est functio transcendens ipsius x.

COROLLARIUM 2

26. Hinc intelligitur, si y fuerit functio transcendens ipsius x, vicissim fore x functionem transcendentem ipsius y atque ex hac conversione novae functiones transcendentes oriuntur.

COROLLARIUM 3

27. Pro variis partibus et sectionibus calculi integralis nascuntur etiam plura genera functionum transcendentium, quorum adeo numerus in infinitum exsurgit; unde patet, quanta copia omnium quantitatum possibilium nobis adhuc sit ignota.

SCHOLION 1

28. Iam antequam in Analysin infinitorum penetravimus, species quasdam functionum transcendentium cognoscere licuit. Primam suppeditavit doctrina logarithmorum; si enim y denotet logarithmum ipsius x, ut sit $y = lx$, erit y utique functio transcendens ipsius x sicque logarithmi quasi primam speciem functionum transcendentium constituunt. Deinde cum ex aequatione $y = lx$ vicissim sit $x = e^y$, erit x utique etiam functio transcendens ipsius y ac tales functiones vocantur exponentiales. Porro autem consideratio angulorum aliud genus aperuit; veluti si angulus, cuius sinus est $= s$, ponatur $= \varphi$, ut sit $\varphi = \text{Arc. sin.}\, s$, nullum est dubium, quin φ sit functio transcendens ipsius s et quidem infinitiformis; hincque cum convertendo prodeat $s = \sin.\varphi$, erit etiam sinus s functio transcendens anguli φ. Quanquam autem hae functiones transcendentes sine subsidio calculi integralis sunt agnitae, tamen in ipso quasi limine calculi integralis ad eas deducimur earumque indoles ita nobis iam est perspecta, ut propemodum functionibus algebraicis accenseri queant. Quare etiam perpetuo in calculo integrali, quoties functiones transcendentes ibi repertas ad logarithmos vel angulos revocare licet, eas tanquam algebraicas spectare solemus.

SCHOLION 2

29. Cum calculus integralis ex inversione calculi differentialis oriatur, perinde ac reliquae methodi inversae ad notitiam novi generis quantitatum nos perducit. Ita si a tirone primorum elementorum nihil praeter notitiam numerorum integrorum positivorum postulemus, apprehensa additione, statim atque ad operationem inversam, subtractionem scilicet, ducitur, notionem numerorum negativorum assequetur. Deinde multiplicatione tradita, cum ad divisionem progreditur, ibi notionem fractionum accipiet. Porro postquam evectionem ad potestates didicerit, si per operationem inversam extractionem radicum suscipiat, quoties negotium non succedit, ideam numerorum irrationalium adipiscetur haecque cognitio per totam Analysin communem sufficiens

censetur. Simili ergo modo calculus integralis, quatenus integratio non succedit, novum nobis genus quantitatum transcendentium aperit. Non enim, uti omnium differentialia exhiberi possunt, ita vicissim omnium differentialium integralia exhibere licet.

SCHOLION 3

30. Neque vero, statim ac primi conatus in integratione expedienda fuerint initi, functiones quaesitae pro transcendentibus sunt habendae; fieri enim saepe solet, ut integrale etiam algebraicum nonnisi per operationes artificiosas obtineri queat. Deinde quando functio quaesita fuerit transcendens, sollicite videndum est, num forte ad species illas simplicissimas logarithmorum vel angulorum revocari possit, quo casu solutio algebraicae esset aequiparanda. Quod si minus successerit, formam tamen simplicissimam functionum transcendentium, ad quam quaesitam reducere liceat, indagari conveniet. Ad usum autem longe commodissimum est, ut valores functionum transcendentium vero proxime exhibeantur, quem in finem insignis pars calculi integralis in investigationem serierum infinitarum impenditur, quae valores earum functionum contineant.

THEOREMA

31. *Omnes functiones per calculum integralem inventae sunt indeterminatae ac requirunt determinationem ex natura quaestionis, cuius solutionem suppeditant, petendam.*

DEMONSTRATIO

Cum semper infinitae dentur functiones, quarum idem est differentiale, siquidem functionis $P+C$, quicunque valor constanti C tribuatur, differentiale idem est $=dP$, vicissim etiam proposito differentiali dP integrale est $P+C$, ubi pro C quantitatem constantem quamcunque ponere licet; unde patet eam functionem, cuius differentiale datur $=dP$, esse indeterminatam, cum quantitatem constantem arbitrariam in se involvat. Idem etiam eveniat necesse est, si functio ex quacunque differentialium relatione sit determinanda, semperque complectetur quantitatem constantem arbitrariam, cuius nullum vestigium in relatione differentialium apparuit. Determinabitur ergo huiusmodi functio per calculum integralem inventa, dum constanti illi arbitrariae certus valor tribuitur, quem semper natura quaestionis, cuius solutio ad illam functionem perduxerat, suppeditabit.

COROLLARIUM 1

32. Si ergo functio y ipsius x ex relatione quapiam differentialium definitur, per constantem arbitrariam ingressam ita determinari potest, ut posito $x = a$ fiat $y = b$; quo facto functio erit determinata et pro quovis valore ipsi x tributo functio y determinatum obtinebit valorem.

COROLLARIUM 2

33. Si ex relatione differentialium secundi gradus functio y definiatur, binas involvet constantes arbitrarias ideoque duplicem determinationem admittit, qua effici potest, ut posito $x = a$ non solum y obtineat datum valorem b, sed etiam ratio $\frac{dy}{dx}$ dato valori c fiat aequalis.

COROLLARIUM 3

34. Si y sit functio binarum variabilium x et t ex relatione differentialium eruta, etiam constantem arbitrariam involvet, cuius determinatione effici poterit, ut posito $t = a$ aequatio inter y et x prodeat data seu naturam datae cuiuspiam curvae exprimat.

SCHOLION

35. Ista functionum integralium, seu quae per calculum integralem sunt inventae, determinatio quovis casu ex natura quaestionis tractatae facile deducitur neque ulla difficultate laborat, nisi forte praeter necessitatem solutio ad differentialia fuerit perducta, cum per Analysin communem erui potuisset; quo casu perinde atque in Algebra quasi radices inutiles ingeruntur. Cum autem haec determinatio tantum in applicatione ad certos casus instituatur, hic, ubi integrandi methodum in genere tradimus, integralia in omni amplitudine eruere conabimur, ita ut constantes per integrationem ingressae maneant arbitrariae, neque, nisi conditio quaedam urgeat, eas determinabimus. Caeterum determinatio functionum ipsius x simplicissima est, qua eae casu $x = 0$ ipsae evanescentes redduntur.

DEFINITIO 6

36. *Integrale completum exhiberi dicitur, quando functio quaesita omni extensione cum constante arbitraria repraesentatur. Quando autem ista constans iam certo modo est determinata, integrale vocari solet particulare.*

COROLLARIUM 1

37. Quovis ergo casu datur unicum integrale completum; integralia autem particularia infinita exhiberi possunt. Sic differentialis xdx integrale completum est $\frac{1}{2}xx + C$, integralia autem particularia $\frac{1}{2}xx$, $\frac{1}{2}xx+1$, $\frac{1}{2}xx+2$ etc. multitudine infinita.

COROLLARIUM 2

38. Integrale ergo completum omnia integralia particularia in se complectitur ex eoque haec omnia facile formari possunt. Vicissim autem ex integralibus particularibus integrale completum non innotescit. Saepenumero autem, uti deinceps patebit, habetur methodus ex integrali particulari completum inveniendi.

SCHOLION

39. Interdum facile est integrale particulare coniectura vel divinatione assequi. Veluti si eiusmodi functio ipsius x, quae sit y, quaeritur, ut sit $dy + yydx = dx + xxdy$, huic aequationi manifesto satisfit sumendo $y = x$, quod ergo est integrale particulare, quoniam in eo nulla inest constans arbitraria; at integrale completum reperietur $y = \frac{1 + Cx}{C + x}$, quod illud particulare in se continet sumendo $C = \infty$. Simili modo sumendo $C = 0$ hinc aliud integrale obtinetur $y = \frac{1}{x}$, quod superiori aequationi perinde satisfacit ac prius $y = x$. Omnia autem integralia particularia, quaecunque satisfaciunt, contineri necesse est in formula generali $y = \frac{1+Cx}{C+x}$, prouti constanti arbitrariae C alii atque alii valores tribuantur; ita sumto $C = 1$ fit etiam $y = 1$. Plerumque autem evenire solet, ut, etiamsi integrale quoddam particulare sit algebraicum, tamen integrale completum sit transcendens. Veluti si proposita sit haec aequatio $dy + ydx = dx + xdx$, statim patet satisfieri posito $y = x$, quod ergo est integrale particulare; verum integrale completum constantem arbitrariam C involvens est $y = x + Ce^{-x}$ denotante e numerum, cuius logarithmus $= 1$; nisi ergo hic sumatur $C = 0$, functio y semper est transcendens.

Haec in genere notasse sufficiat, antequam ad tractationem ipsam calculi integralis aggrediamur, quandoquidem ad omnes integrationes pertinent; nunc igitur forma tractationis exposita ad opus tractandum pergamus.

CONSPECTVS
VNIVERSI OPERIS
DE
CALCVLO INTEGRALI.

LIBER PRIOR : tradit methodum inueftigandi functiones vnius variabilis ex data quadam relatione differentialium, continetque duas partes :

Pars prior: quando relatio illa data tantum differentialia primi gradus complectitur.

Pars pofterior: quando relatio illa data differentialia fecundi altiorumue graduum complectitur.

LIBER POSTERIOR : tradit methodum inueftigandi functiones duarum pluriumue variabilium ex data quadam relatione differentialium , continetque duas partes :

Pars prior: quando relatio illa data tantum differentialia primi gradus complectitur.

Pars pofterior: quando relatio illa data differentialia fecundi altiorumue graduum complectitur.

CALCVLI INTEGRALIS.
LIBER PRIOR.

PARS PRIMA
SEV
METHODVS INVESTIGANDI FVNCTIONES
VNIVS VARIABILIS EX DATA RELATIONE QVACVNQVE DIFFERENTIALIVM PRIMI GRADVS.

SECTIO PRIMA
DE
INTEGRATIONE FORMVLARVM
DIFFERENTIALIVM.

CAPUT I

DE INTEGRATIONE FORMULARUM DIFFERENTIALIUM RATIONALIUM

DEFINITIO

40. *Formula differentialis* rationalis *est, quando variabilis x, cuius functio quaeritur, differentiale dx multiplicatur in functionem rationalem ipsius x, seu si X designet functionem rationalem ipsius x, haec formula differentialis Xdx dicitur rationalis.*

COROLLARIUM 1

41. In hoc ergo capite eiusmodi functio ipsius x quaeritur, quae si ponatur y, ut $\frac{dy}{dx}$ aequetur functioni rationali ipsius x, seu posita tali functione $= X$ ut sit $\frac{dy}{dx} = X$.

COROLLARIUM 2

42. Hinc quaeritur eiusmodi functio ipsius x, cuius differentiale sit $= Xdx$; huius ergo integrale, quod ita indicari solet $\int Xdx$, praebebit functionem quaesitam.

COROLLARIUM 3

43. Quodsi P fuerit eiusmodi functio ipsius x, ut eius differentiale dP sit $= Xdx$, quoniam quantitatis $P + C$ idem est differentiale, formulae propositae Xdx integrale completum est $P + C$.

SCHOLION 1

44. Ad libri primi partem priorem huiusmodi referuntur quaestiones, quibus functiones solius variabilis x ex data differentialium primi gradus relatione quaeruntur. Scilicet si functio quaesita $= y$ et $\frac{dy}{dx} = p$, id praestari oportet, ut proposita aequatione quacunque inter ternas quantitates x, y et p inde indoles functionis y seu aequatio inter x et y, elisa littera p, inveniatur. Quaestio autem sic in genere proposita vires Analyseos adeo superare videtur, ut eius solutio nunquam expectari queat. In casibus igitur simplicioribus vires nostrae sunt exercendae, inter quos primum occurrit casus, quo p functioni cuipiam ipsius x, puta X, aequatur, ut sit $\frac{dy}{dx} = X$ seu $dy = Xdx$ ideoque integrale $y = \int Xdx$ requiratur, in quo primam sectionem collocamus. Verum et hic casus pro varia indole functionis X latissime patet ac plurimis difficultatibus implicatur, unde in hoc capite eiusmodi tantum quaestiones evolvere instituimus, in quibus ista functio X est rationalis, deinceps ad functiones irrationales atque adeo transcendentes progressuri. Hinc ista pars commode in duas sectiones subdividitur, in quarum altera integratio formularum simplicium, quibus $p = \frac{dy}{dx}$ functioni tantum ipsius x aequatur, est tradenda, in altera autem rationem integrandi doceri conveniet, cum proposita fuerit aequatio quaecunque ipsarum x, y et p. Et cum in his duabus sectionibus ac potissimum priore a Geometris plurimum sit elaboratum, eae fere maximam partem totius operis complebunt.

SCHOLION 2

45. Prima autem integrationis principia ex ipso calculo differentiali sunt petenda, perinde ac principia divisionis ex multiplicatione et principia extractionis radicum ex ratione evectionis ad potestates sumi solent. Cum igitur, si quantitas differentianda ex pluribus partibus constet ut $P + Q - R$, eius differentiale sit $dP + dQ - dR$, ita vicissim, si formula differentialis ex pluribus partibus constet ut $Pdx + Qdx - Rdx$, integrale erit

$$\int Pdx + \int Qdx - \int Rdx$$

singulis scilicet partibus seorsim integrandis. Deinde cum quantitatis aP differentiale sit adP, formulae differentialis $aPdx$ integrale erit $a\int Pdx$; scilicet per quam quantitatem constantem formula differentialis multiplicatur, per eandem integrale multiplicari debet. Ita si formula differentialis sit $aPdx + bQdx + cRdx$, quaecunque functiones ipsius x litteris P, Q, R designentur, integrale erit

$$a\int Pdx + b\int Qdx + c\int Rdx,$$

ita ut integratio tantum in singulis formulis Pdx, Qdx et Rdx sit instituenda, hocque facto insuper adiici debet constans arbitraria C, ut integrale completum obtineatur.

PROBLEMA 1

46. *Invenire functionem ipsius x, ut eius differentiale sit $= ax^n dx$, seu integrare formulam differentialem $ax^n dx$.*

SOLUTIO

Cum potestatis x^m differentiale sit $mx^{m-1}dx$, erit vicissim

$$\int mx^{m-1}dx = m\int x^{m-1}dx = x^m$$

ideoque

$$\int x^{m-1}dx = \frac{1}{m}x^m;$$

fiat $m - 1 = n$ seu $m = n + 1$; erit

$$\int x^n dx = \frac{1}{n+1}x^{n+1} \quad \text{et} \quad a\int x^n dx = \frac{a}{n+1}x^{n+1}.$$

Unde formulae differentialis propositae $ax^n dx$ integrale completum erit

$$\frac{a}{n+1}x^{n+1} + C,$$

cuius ratio vel inde patet, quod eius differentiale revera fit $= ax^n dx$. Atque haec integratio semper locum habet, quicunque numerus exponenti n tribuatur, sive positivus sive negativus, sive integer sive fractus, sive etiam irrationalis.

Unicus casus hinc excipitur, quo est exponens $n = -1$ seu haec formula $\frac{adx}{x}$ integranda proponitur. Verum in calculo differentiali iam ostendimus, si lx denotet logarithmum hyperbolicum ipsius x, fore eius differentiale $=\frac{dx}{x}$, unde vicissim concludimus esse

$$\int\frac{dx}{x}=lx \quad \text{et} \quad \int\frac{adx}{x}=alx.$$

Quare adiecta constante arbitraria erit formulae $\frac{adx}{x}$ integrale completum

$$=alx+C=lx^a+C,$$

quod etiam pro C ponendo lc ita exprimitur: lcx^a.

COROLLARIUM 1

47. Formulae ergo differentialis $ax^n dx$ integrale semper est algebraicum solo excepto casu, quo $n=-1$ et integrale per logarithmos exprimitur, qui ad functiones transcendentes sunt referendi. Est scilicet

$$\int\frac{adx}{x}=alx+C=lcx^a.$$

COROLLARIUM 2

48. Si exponens n numeros positivos denotet, sequentes integrationes utpote maxime obviae probe sunt tenendae

$$\int adx=ax+C,\quad \int axdx=\frac{a}{2}xx+C,\quad \int ax^2dx=\frac{a}{3}x^3+C,$$

$$\int ax^3dx=\frac{a}{4}x^4+C,\quad \int ax^4dx=\frac{a}{5}x^5+C,\quad \int ax^5dx=\frac{a}{6}x^6+C.$$

COROLLARIUM 3

49. Si n sit numerus negativus, posito $n=-m$ fit

$$\int\frac{adx}{x^m}=\frac{a}{1-m}x^{1-m}+C=\frac{-a}{(m-1)x^{m-1}}+C;$$

unde hi casus simpliciores notentur

$$\int\frac{adx}{x^2}=\frac{-a}{x}+C,\quad \int\frac{adx}{x^3}=\frac{-a}{2xx}+C,\quad \int\frac{adx}{x^4}=\frac{-a}{3x^3}+C,$$

$$\int\frac{adx}{x^5}=\frac{-a}{4x^4}+C,\quad \int\frac{adx}{x^6}=\frac{-a}{5x^5}+C \quad \text{etc.}$$

COROLLARIUM 4

50. Quin etiam si n denotet numeros fractos, integralia hinc obtinentur. Sit primo $n=\frac{m}{2}$; erit

$$\int a\,dx\sqrt{x^m}=\frac{2a}{m+2}x\sqrt{x^m}+C,$$

unde casus notentur

$$\int a\,dx\sqrt{x}=\frac{2a}{3}x\sqrt{x}+C,\quad \int ax\,dx\sqrt{x}=\frac{2a}{5}x^2\sqrt{x}+C,$$

$$\int axx\,dx\sqrt{x}=\frac{2a}{7}x^3\sqrt{x}+C,\quad \int ax^3\,dx\sqrt{x}=\frac{2a}{9}x^4\sqrt{x}+C.$$

COROLLARIUM 5

51. Ponatur etiam $n=\frac{-m}{2}$ et habebitur

$$\int\frac{a\,dx}{\sqrt{x^m}}=\frac{2a}{2-m}\frac{x}{\sqrt{x^m}}+C=\frac{-2a}{(m-2)\sqrt{x^{m-2}}}+C,$$

unde hi casus notentur

$$\int\frac{a\,dx}{\sqrt{x}}=2a\sqrt{x}+C,\quad \int\frac{a\,dx}{x\sqrt{x}}=\frac{-2a}{\sqrt{x}}+C,\quad \int\frac{a\,dx}{xx\sqrt{x}}=\frac{-2a}{3x\sqrt{x}}+C,$$

$$\int\frac{a\,dx}{x^3\sqrt{x}}=\frac{-2a}{5x^2\sqrt{x}}+C.$$

COROLLARIUM 6

52. Si in genere ponamus $n=\frac{\mu}{\nu}$, fiet

$$\int ax^{\frac{\mu}{\nu}}dx=\frac{\nu a}{\mu+\nu}x^{\frac{\mu+\nu}{\nu}}+C$$

seu per radicalia

$$\int a\,dx\sqrt[\nu]{x^\mu}=\frac{\nu a}{\mu+\nu}\sqrt[\nu]{x^{\mu+\nu}}+C;$$

sin autem ponatur $n=\frac{-\mu}{\nu}$, habebitur

$$\int \frac{a\,dx}{x^{\frac{\mu}{\nu}}} = \frac{\nu a}{\nu-\mu} x^{\frac{\nu-\mu}{\nu}} + C$$

seu per radicalia

$$\int \frac{a\,dx}{\sqrt[\nu]{x^{\mu}}} = \frac{\nu a}{\nu-\mu} \sqrt[\nu]{x^{\nu-\mu}} + C.$$

SCHOLION 1

53. Quanquam in hoc capite functiones tantum rationales tractare institueram, tamen istae irrationalitates tam sponte se obtulerunt, ut perinde ac rationales tractari possint. Caeterum hinc quoque formulae magis complicatae integrari possunt, si pro x functiones alius cuiuspiam variabilis z statuantur. Veluti si ponamus $x = f + gz$, erit $dx = g\,dz$; quare si pro a scribamus $\frac{a}{g}$, habebitur

$$\int a\,dz(f+gz)^n = \frac{a}{(n+1)g}(f+gz)^{n+1} + C,$$

casu autem singulari, quo $n = -1$,

$$\int \frac{a\,dz}{f+gz} = \frac{a}{g} l(f+gz) + C.$$

Tum si sit $n = -m$, fiet

$$\int \frac{a\,dz}{(f+gz)^m} = \frac{-a}{(m-1)g(f+gz)^{m-1}} + C.$$

At posito $n = \frac{\mu}{\nu}$ prodit

$$\int a\,dz(f+gz)^{\frac{\mu}{\nu}} = \frac{\nu a}{(\nu+\mu)g}(f+gz)^{\frac{\mu}{\nu}+1} + C;$$

posito autem $n = -\frac{\mu}{\nu}$ obtinetur

$$\int \frac{a\,dz}{(f+gz)^{\frac{\mu}{\nu}}} = \frac{\nu a(f+gz)}{(\nu-\mu)g(f+gz)^{\frac{\mu}{\nu}}} + C.$$

SCHOLION 2

54. Caeterum hic insignis proprietas annotari meretur. Cum hic quaeratur functio y, ut sit $dy = ax^n dx$, si ponamus $\frac{dy}{dx} = p$, haec habebitur relatio

$p = ax^n$, ex qua functio y investigari debet. Quoniam igitur est

$$y = \frac{a}{n+1} x^{n+1} + C,$$

ob $ax^n = p$ erit quoque

$$y = \frac{px}{n+1} + C$$

sicque casum habemus, ubi relatio differentialium per aequationem quandam inter x, y et p proponitur cuique iam novimus satisfieri per aequationem $y = \frac{a}{n+1} x^{n+1} + C$. Verum haec non amplius erit integrale completum pro relatione in aequatione $y = \frac{px}{n+1} + C$ contenta, sed tantum particulare, quoniam integrale illud non involvit novam constantem, quae in relatione differentiali non insit. Integrale autem completum est

$$y = \frac{aD}{n+1} x^{n+1} + C$$

novam constantem D involvens; hinc enim fit

$$\frac{dy}{dx} = aDx^n = p \quad \text{ideoque} \quad y = \frac{px}{n+1} + C.$$

Etsi hoc non ad praesens institutum pertinet, tamen notasse iuvabit.

PROBLEMA 2

55. *Invenire functionem ipsius x, cuius differentiale sit $= Xdx$ denotante X functionem quamcunque rationalem integram ipsius x, seu definire integrale $\int Xdx$.*

SOLUTIO

Cum X sit functio rationalis integra ipsius x, in hac forma contineatur necesse est

$$X = \alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \zeta x^5 + \text{etc.},$$

unde per problema praecedens integrale quaesitum est

$$\int Xdx = C + \alpha x + \frac{1}{2}\beta x^2 + \frac{1}{3}\gamma x^3 + \frac{1}{4}\delta x^4 + \frac{1}{5}\varepsilon x^5 + \frac{1}{6}\zeta x^6 + \text{etc.}$$

Atque in genere si sit

$$X = \alpha x^{\lambda} + \beta x^{\mu} + \gamma x^{\nu} + \text{etc.},$$

erit

$$\int X dx = C + \frac{\alpha}{\lambda+1} x^{\lambda+1} + \frac{\beta}{\mu+1} x^{\mu+1} + \frac{\gamma}{\nu+1} x^{\nu+1} + \text{etc.},$$

ubi exponentes λ, μ, ν etc. etiam numeros tam negativos quam fractos significare possunt, dummodo notetur, si fuerit $\lambda = -1$, fore $\int \frac{\alpha dx}{x} = \alpha lx$, qui est unicus casus ad ordinem transcendentium referendus.

PROBLEMA 3

56. *Si X denotet functionem quamcunque rationalem fractam ipsius x, methodum describere, cuius ope formulae $X dx$ integrale investigari conveniat.*

SOLUTIO

Sit igitur $X = \frac{M}{N}$, ita ut M et N futurae sint functiones integrae ipsius x, ac primo dispiciatur, num summa potestas ipsius x in numeratore M tanta sit vel etiam maior quam in denominatore N, quo casu ex fractione $\frac{M}{N}$ partes integrae per divisionem eliciantur; quarum integratio cum nihil habeat difficultatis, totum negotium reducitur ad eiusmodi fractionem $\frac{M}{N}$, in cuius numeratore M summa potestas ipsius x minor sit quam in denominatore N.

Tum quaerantur omnes factores ipsius denominatoris N, tam simplices, si fuerint reales, quam duplices reales, vicem scilicet binorum simplicium imaginariorum gerentes; simulque videndum est, utrum hi factores omnes sint inaequales necne; pro factorum enim aequalitate alio modo resolutio fractionis $\frac{M}{N}$ in fractiones simplices est instituenda, quandoquidem ex singulis factoribus fractiones partiales nascuntur, quarum aggregatum fractioni propositae $\frac{M}{N}$ aequatur. Scilicet ex factore simplici $a + bx$ nascitur fractio

$$\frac{A}{a+bx};$$

si bini sint aequales seu denominator N factorem habeat $(a + bx)^2$, hinc nascuntur fractiones

$$\frac{A}{(a+bx)^2} + \frac{B}{a+bx};$$

ex huiusmodi autem factore $(a + bx)^3$ hae tres fractiones

$$\frac{A}{(a+bx)^3} + \frac{B}{(a+bx)^2} + \frac{C}{a+bx}$$

et ita porro.

Factor autem duplex, cuius forma est $aa - 2abx\cos.\zeta + bbxx$, nisi alius ipsi fuerit aequalis, dabit fractionem partialem

$$\frac{A+Bx}{aa-2abx\cos.\zeta+bbxx};$$

si autem denominator N duos huiusmodi factores aequales involvat, inde nascuntur binae huiusmodi fractiones partiales

$$\frac{A+Bx}{(aa-2abx\cos.\zeta+bbxx)^2} + \frac{C+Dx}{aa-2abx\cos.\zeta+bbxx};$$

at si cubus adeo $(aa - 2abx\cos.\zeta + bbxx)^3$ fuerit factor denominatoris N, ex eo oriuntur huiusmodi tres fractiones partiales

$$\frac{A+Bx}{(aa-2abx\cos.\zeta+bbxx)^3} + \frac{C+Dx}{(aa-2abx\cos.\zeta+bbxx)^2} + \frac{E+Fx}{aa-2abx\cos.\zeta+bbxx}$$

et ita porro.

Cum igitur hoc modo fractio proposita $\frac{M}{N}$ in omnes suas fractiones simplices fuerit resoluta, omnes continebuntur in alterutra harum formarum vel

$$\frac{A}{(a+bx)^n} \quad \text{vel} \quad \frac{A+Bx}{(aa-2abx\cos.\zeta+bbxx)^n}$$

ac singulos iam per dx multiplicatos integrari oportet; erit omnium horum integralium aggregatum valor functionis quaesitae $\int X dx = \int \frac{M}{N} dx$.

COROLLARIUM 1

57. Pro integratione ergo omnium huiusmodi formularum $\frac{M}{N}dx$ totum negotium reducitur ad integrationem huiusmodi binarum formularum

$$\int \frac{A\,dx}{(a+bx)^n} \quad \text{et} \quad \int \frac{(A+Bx)\,dx}{(aa-2abx\cos.\zeta+bbxx)^n},$$

dum pro n successive scribuntur numeri 1, 2, 3, 4 etc.

COROLLARIUM 2

58. Ac prioris quidem formae integrale iam supra (§ 53) est expeditum, unde patet fore

$$\int \frac{A\,dx}{a+bx} = \frac{A}{b}\,l(a+bx) + \text{Const.}$$

$$\int \frac{A\,dx}{(a+bx)^2} = \frac{-A}{b(a+bx)} + \text{Const.}$$

$$\int \frac{A\,dx}{(a+bx)^3} = \frac{-A}{2b(a+bx)^2} + \text{Const.}$$

et generatim

$$\int \frac{A\,dx}{(a+bx)^n} = \frac{-A}{(n-1)b(a+bx)^{n-1}} + \text{Const.}$$

COROLLARIUM 3

59. Ad propositum ergo absolvendum nihil aliud superest, nisi ut integratio huius formulae

$$\int \frac{(A+Bx)\,dx}{(aa-2abx\cos.\zeta+bbxx)^n}$$

doceatur, primo quidem casu $n=1$, tum vero casibus $n=2$, $n=3$, $n=4$ etc.

SCHOLION 1

60. Nisi vellemus imaginaria evitare, totum negotium ex iam traditis confici posset; denominatore enim N in omnes suos factores simplices resoluto, sive sint reales sive imaginarii, fractio proposita semper resolvi poterit in fractiones partiales huius formae $\frac{A}{a+bx}$ vel huius $\frac{A}{(a+bx)^n}$; quarum integralia cum sint in promtu, totius formae $\frac{M}{N}dx$ integrale habetur. Tum autem non parum molestum foret binas partes imaginarias ita coniungere, ut expressio realis resultaret, quod tamen rei natura absolute exigit.

SCHOLION 2

61. Hic utique postulamus resolutionem cuiusque functionis integrae in factores nobis concedi, etiamsi algebra neutiquam adhuc eo sit perducta, ut haec resolutio actu institui possit. Hoc autem in Analysi ubique postulari

solet, ut, quo longius progrediamur, ea, quae retro sunt relicta, etiamsi non satis fuerint explorata, tanquam cognita assumamus; sufficere scilicet hic potest omnes factores per methodum approximationum quantumvis prope assignari posse. Simili modo cum in calculo integrali longius processerimus, integralia omnium huiusmodi formularum Xdx, quaecunque functio ipsius x littera X significetur, tanquam cognita spectabimus plurimumque nobis praestitisse videbimur, si integralia magis abscondita ad eas formas reducere valuerimus; atque hoc etiam in usu practico nihil turbat, cum valores talium formularum $\int Xdx$ quantumvis prope assignare liceat, uti in sequentibus ostendemus. Caeterum ad has integrationes resolutio denominatoris N in suos factores absolute est necessaria, propterea quod singuli hi factores in expressionem integralis ingrediuntur; paucissimi sunt casus iique maxime obvii, quibus ista resolutione carere possumus; veluti si proponatur haec formula $\frac{x^{n-1}dx}{1+x^n}$, statim patet posito $x^n = v$ eam abire in $\frac{dv}{n(1+v)}$, cuius integrale est $\frac{1}{n}l(1+v) = \frac{1}{n}l(1+x^n)$, ubi resolutione in factores non fuerat opus. Verum huiusmodi casus per se tam sunt perspicui, ut eorum tractatio nulla peculiari explicatione indigeat.

PROBLEMA 4

62. *Invenire integrale huius formulae*

$$y = \int \frac{(A+Bx)dx}{aa - 2abx\cos.\zeta + bbxx}.$$

SOLUTIO

Cum numerator duabus constet partibus $Adx + Bxdx$, haec posterior $Bxdx$ sequenti modi tolli poterit. Cum sit

$$l(aa - 2abx\cos.\zeta + bbxx) = \int \frac{-2abdx\cos.\zeta + 2bbxdx}{aa - 2abx\cos.\zeta + bbxx},$$

multiplicetur haec aequatio per $\frac{B}{2bb}$ et a proposita auferatur; sic enim prodibit

$$y - \frac{B}{2bb}l(aa - 2abx\cos.\zeta + bbxx) = \int \frac{\left(A + \frac{Ba\cos.\zeta}{b}\right)dx}{aa - 2abx\cos.\zeta + bbxx},$$

ita ut haec tantum formula integranda supersit. Ponatur brevitatis gratia

$$A + \frac{Ba\cos.\zeta}{b} = C,$$

ut habeatur haec formula

$$\int \frac{Cdx}{aa - 2abx\cos.\zeta + bbxx},$$

quae ita exhiberi potest

$$\int \frac{Cdx}{aa\sin.\zeta^2 + (bx - a\cos.\zeta)^2}.$$

Statuatur $bx - a\cos.\zeta = av\sin.\zeta$ hincque $dx = \frac{adv\sin.\zeta}{b}$, unde formula nostra erit

$$\int \frac{Cadv\sin.\zeta : b}{aa\sin.\zeta^2(1+vv)} = \frac{C}{ab\sin.\zeta}\int \frac{dv}{1+vv}.$$

Ex calculo autem differentiali novimus esse

$$\int \frac{dv}{1+vv} = \text{Arc. tang.}\, v = \text{Arc. tang.}\, \frac{bx - a\cos.\zeta}{a\sin.\zeta},$$

unde ob

$$C = \frac{Ab + Ba\cos.\zeta}{b}$$

erit nostrum integrale

$$\frac{Ab + Ba\cos.\zeta}{abb\sin.\zeta}\,\text{Arc. tang.}\,\frac{bx - a\cos.\zeta}{a\sin.\zeta}.$$

Quocirca formulae propositae

$$\frac{(A + Bx)dx}{aa - 2abx\cos.\zeta + bbxx}$$

integrale est

$$\frac{B}{2bb}l(aa - 2abx\cos.\zeta + bbxx) + \frac{Ab + Ba\cos.\zeta}{abb\sin.\zeta}\,\text{Arc. tang.}\,\frac{bx - a\cos.\zeta}{a\sin.\zeta},$$

quod ut fiat completum, constans arbitraria C insuper addatur.

COROLLARIUM 1

63. Si ad Arc. tang. $\frac{bx - a\cos.\zeta}{a\sin.\zeta}$ addamus Arc. tang. $\frac{\cos.\zeta}{\sin.\zeta}$, quippe qui in constante addenda contentus concipiatur, prodibit Arc. tang. $\frac{bx\sin.\zeta}{a - bx\cos.\zeta}$ sicque

habebimus

$$\int \frac{(A+Bx)dx}{aa-2abx\cos.\zeta+bbxx}$$

$$=\frac{B}{2bb}l(aa-2abx\cos.\zeta+bbxx)+\frac{Ab+Ba\cos.\zeta}{abb\sin.\zeta}\text{Arc. tang.}\frac{bx\sin.\zeta}{a-bx\cos.\zeta}$$

adiecta constante C.

COROLLARIUM 2

64. Si velimus, ut integrale hoc evanescat posito $x=0$, constans C sumi debet $=\frac{-B}{2bb}laa$ sicque fiet

$$\int \frac{(A+Bx)dx}{aa-2abx\cos.\zeta+bbxx}$$

$$=\frac{B}{bb}l\frac{\sqrt{(aa-2abx\cos.\zeta+bbxx)}}{a}+\frac{Ab+Ba\cos.\zeta}{abb\sin.\zeta}\text{Arc. tang.}\frac{bx\sin.\zeta}{a-bx\cos.\zeta}.$$

Pendet ergo hoc integrale partim a logarithmis, partim ab arcubus circularibus seu angulis.

COROLLARIUM 3

65. Si littera B evanescat, pars a logarithmis pendens evanescit fitque

$$\int \frac{Adx}{aa-2abx\cos.\zeta+bbxx}=\frac{A}{ab\sin.\zeta}\text{Arc. tang.}\frac{bx\sin.\zeta}{a-bx\cos.\zeta}+C$$

sicque per solum angulum definitur.

COROLLARIUM 4

66. Si angulus ζ sit rectus ideoque $\cos.\zeta=0$ et $\sin.\zeta=1$, habebitur

$$\int \frac{(A+Bx)dx}{aa+bbxx}=\frac{B}{bb}l\frac{\sqrt{(aa+bbxx)}}{a}+\frac{A}{ab}\text{Arc. tang.}\frac{bx}{a}+C;$$

si angulus ζ sit 60^0 ideoque $\cos.\zeta=\frac{1}{2}$ et $\sin.\zeta=\frac{\sqrt{3}}{2}$, erit

$$\int \frac{(A+Bx)dx}{aa-abx+bbxx}=\frac{B}{bb}l\frac{\sqrt{(aa-abx+bbxx)}}{a}+\frac{2Ab+Ba}{abb\sqrt{3}}\text{Arc. tang.}\frac{bx\sqrt{3}}{2a-bx}.$$

At si $\zeta = 120^0$ ideoque $\cos.\zeta = -\frac{1}{2}$ et $\sin.\zeta = \frac{\sqrt{3}}{2}$, erit

$$\int \frac{(A+Bx)dx}{aa+abx+bbxx} = \frac{B}{bb} l \frac{\sqrt{(aa+abx+bbxx)}}{a} + \frac{2Ab-Ba}{abb\sqrt{3}} \text{Arc. tang.} \frac{bx\sqrt{3}}{2a+bx}.$$

SCHOLION 1

67. Omnino hic notatu dignum evenit, quod casu $\zeta = 0$, quo denominator $aa - 2abx + bbxx$ fit quadratum, ratio anguli ex integrali discedat. Posito enim angulo ζ infinite parvo erit $\cos.\zeta = 1$ et $\sin.\zeta = \zeta$; unde pars logarithmica fit $\frac{B}{bb} l \frac{a-bx}{a}$ et altera pars $\frac{Ab+Ba}{abb\zeta}$ Arc. tang. $\frac{bx\zeta}{a-bx} = \frac{(Ab+Ba)x}{ab(a-bx)}$, quia arcus infinite parvi $\frac{bx\zeta}{a-bx}$ tangens ipsi est aequalis, sicque haec pars fit algebraica. Quocirca erit

$$\int \frac{(A+Bx)dx}{(a-bx)^2} = \frac{B}{bb} l \frac{a-bx}{a} + \frac{(Ab+Ba)x}{ab(a-bx)} + \text{Const.},$$

cuius veritas ex praecedentibus est manifesta; est enim

$$\frac{A+Bx}{(a-bx)^2} = \frac{-B}{b(a-bx)} + \frac{Ab+Ba}{b(a-bx)^2}.$$

Iam vero est

$$\int \frac{-Bdx}{b(a-bx)} = \frac{B}{bb} l(a-bx) - \frac{B}{bb} la = \frac{B}{bb} l \frac{a-bx}{a},$$

$$\int \frac{(Ab+Ba)dx}{b(a-bx)^2} = \frac{Ab+Ba}{bb(a-bx)} - \frac{Ab+Ba}{abb} = \frac{(Ab+Ba)x}{ab(a-bx)},$$

siquidem utraque integratio ita determinetur, ut casu $x = 0$ integralia evanescant.

SCHOLION 2

68. Simili modo, quo hic usi sumus, si in formula differentiali fracta $\frac{Mdx}{N}$ summa potestas ipsius x in numeratore M uno gradu minor sit quam in denominatore N, etiam is terminus tolli poterit. Sit enim

$$M = Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \text{etc.}$$

et

$$N = \alpha x^n + \beta x^{n-1} + \gamma x^{n-2} + \text{etc.}$$

ac ponatur $\frac{Mdx}{N} = dy$. Cum iam sit

$$dN = n\alpha x^{n-1}dx + (n-1)\beta x^{n-2}dx + (n-2)\gamma x^{n-3}dx + \text{etc.},$$

erit

$$\frac{AdN}{n\alpha N} = \frac{dx}{N}\left(Ax^{n-1} + \frac{(n-1)A\beta}{n\alpha}x^{n-2} + \frac{(n-2)A\gamma}{n\alpha}x^{n-3} + \text{etc.}\right),$$

quo valore inde subtracto remanebit

$$dy - \frac{AdN}{n\alpha N} = \frac{dx}{N}\left(\left(B - \frac{(n-1)A\beta}{n\alpha}\right)x^{n-2} + \left(C - \frac{(n-2)A\gamma}{n\alpha}\right)x^{n-3} + \text{etc.}\right).$$

Quare si brevitatis gratia ponatur

$$B - \frac{(n-1)A\beta}{n\alpha} = \mathfrak{B}, \quad C - \frac{(n-2)A\gamma}{n\alpha} = \mathfrak{C}, \quad D - \frac{(n-3)A\delta}{nn} = \mathfrak{D} \text{ etc.},$$

obtinebitur

$$y = \frac{A}{n\alpha}lN + \int \frac{dx(\mathfrak{B}x^{n-2} + \mathfrak{C}x^{n-3} + \mathfrak{D}x^{n-4} + \text{etc.})}{\alpha x^n + \beta x^{n-1} + \gamma x^{n-2} + \delta x^{n-3} + \text{etc.}} = \int \frac{Mdx}{N}.$$

Hoc igitur modo omnes formulae differentiales fractae eo reduci possunt, ut summa potestas ipsius x in numeratore duobus pluribusve gradibus minor sit quam in denominatore.

PROBLEMA 5

69. *Formulam integralem*

$$\int \frac{(A + Bx)dx}{(aa - 2abx\cos.\zeta + bbxx)^{n+1}}$$

ad aliam similem reducere, ubi potestas denominatoris sit uno gradu inferior.

SOLUTIO

Sit brevitatis gratia $aa - 2abx\cos.\zeta + bbxx = X$ ac ponatur

$$\int \frac{(A + Bx)\,dx}{X^{n+1}} = y.$$

Cum ob
$$dX = -2abdx\cos.\zeta + 2bbxdx$$
sit
$$d.\frac{C+Dx}{X^n} = \frac{-n(C+Dx)dX}{X^{n+1}} + \frac{Ddx}{X^n}$$
ideoque
$$\frac{C+Dx}{X^n} = \int \frac{2nb(C+Dx)(a\cos.\zeta - bx)dx}{X^{n+1}} + \int \frac{Ddx}{X^n},$$
habebimus
$$y + \frac{C+Dx}{X^n} = \int \frac{dx(A+2nCab\cos.\zeta + x(B+2nDab\cos.\zeta - 2nCbb) - 2nDbbxx)}{X^{n+1}} + \int \frac{Ddx}{X^n}.$$

Iam in formula priori litterae C et D ita definiantur, ut numerator per X fiat divisibilis; oportet ergo sit $= -2nDXdx$, unde nanciscimur
$$A + 2nCab\cos.\zeta = -2nDaa$$
et
$$B + 2nDab\cos.\zeta - 2nCbb = 4nDab\cos.\zeta$$
seu $B - 2nCbb = 2nDab\cos.\zeta$ hincque
$$2nDa = \frac{B-2nCbb}{b\cos.\zeta};$$
at ex priori conditione est
$$2nDa = \frac{-A-2nCab\cos.\zeta}{a},$$
quibus aequatis fit
$$Ba + Ab\cos.\zeta - 2nCabb\sin.\zeta^2 = 0$$
seu
$$C = \frac{Ba+Ab\cos.\zeta}{2nabb\sin.\zeta^2},$$
unde
$$B - 2nCbb = \frac{Ba\sin.\zeta^2 - Ba - Ab\cos.\zeta}{a\sin.\zeta^2} = \frac{-Ab\cos.\zeta - Ba\cos.\zeta^2}{a\sin.\zeta^2},$$
ita ut reperiatur
$$D = \frac{-Ab-Ba\cos.\zeta}{2naab\sin.\zeta^2}.$$
Sumtis ergo litteris
$$C = \frac{Ba+Ab\cos.\zeta}{2nabb\sin.\zeta^2} \quad \text{et} \quad D = \frac{-Ab-Ba\cos.\zeta}{2naab\sin.\zeta^2}$$

erit

$$y + \frac{C + Dx}{X^n} = \int \frac{-2nDdx}{X^n} + \int \frac{Ddx}{X^n} = -(2n-1)D\int \frac{dx}{X^n}$$

ideoque

$$\int \frac{(A+Bx)dx}{X^{n+1}} = \frac{-C-Dx}{X^n} - (2n-1)D\int \frac{dx}{X^n},$$

sive

$$\int \frac{(A+Bx)dx}{X^{n+1}} = \frac{-Baa - Aab\cos.\zeta + (Abb + Bab\cos.\zeta)x}{2naabb\sin.\zeta^2 X^n}$$

$$+ \frac{(2n-1)(Ab + Ba\cos.\zeta)}{2naab\sin.\zeta^2}\int \frac{dx}{X^n}.$$

Quare, si formula $\int \frac{dx}{X^n}$ constet, etiam integrale hoc $\int \frac{(A+Bx)dx}{X^{n+1}}$ assignari poterit.

COROLLARIUM 1

70. Cum igitur manente $X = aa - 2abx\cos.\zeta + bbxx$ sit

$$\int \frac{dx}{X} = \frac{1}{ab\sin.\zeta}\text{Arc. tang.}\frac{bx\sin.\zeta}{a - bx\cos.\zeta} + \text{Const.},$$

erit

$$\int \frac{(A+Bx)dx}{X^2} = \frac{-Baa - Aab\cos.\zeta + (Abb + Bab\cos.\zeta)x}{2aabb\sin.\zeta^2 X}$$

$$+ \frac{Ab + Ba\cos.\zeta}{2a^3bb\sin.\zeta^3}\text{Arc. tang.}\frac{bx\sin.\zeta}{a - bx\cos.\zeta} + \text{Const.}$$

Ideoque posito $B = 0$ et $A = 1$ fiet

$$\int \frac{dx}{X^2} = \frac{-a\cos.\zeta + bx}{2aab\sin.\zeta^2 X} + \frac{1}{2a^3b\sin.\zeta^3}\text{Arc. tang.}\frac{bx\sin.\zeta}{a - bx\cos.\zeta} + \text{Const.}$$

Integrale ergo $\int \frac{(A+Bx)dx}{X^2}$ logarithmos non involvit.

COROLLARIUM 2

71. Hinc ergo cum sit

$$\int \frac{dx}{X^3} = \frac{-a\cos.\zeta + bx}{4aab\sin.\zeta^2 X^2} + \frac{3}{4aa\sin.\zeta^2}\int \frac{dx}{X^2} + \text{Const.},$$

erit illum valorem substituendo

$$\int\frac{dx}{X^3}=\frac{-a\cos.\zeta+bx}{4aab\sin.\zeta^2X^2}+\frac{3(-a\cos.\zeta+bx)}{2\cdot 4a^4b\sin.\zeta^4X}+\frac{1\cdot 3}{2\cdot 4a^5b\sin.\zeta^5}\text{Arc. tang.}\frac{bx\sin.\zeta}{a-bx\cos.\zeta}$$

hincque porro concluditur

$$\int\frac{dx}{X^4}=\frac{-a\cos.\zeta+bx}{6aab\sin.\zeta^2X^3}+\frac{5(-a\cos.\zeta+bx)}{4\cdot 6a^4b\sin.\zeta^4X^2}+\frac{3\cdot 5(-a\cos.\zeta+bx)}{2\cdot 4\cdot 6a^6b\sin.\zeta^6X}$$
$$+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6a^7b\sin.\zeta^7}\text{Arc. tang.}\frac{bx\sin.\zeta}{a-bx\cos.\zeta}.$$

COROLLARIUM 3

72. Sic ulterius progrediendo omnium huiusmodi formularum integralia obtinebuntur

$$\int\frac{dx}{X},\quad\int\frac{dx}{X^2},\quad\int\frac{dx}{X^3},\quad\int\frac{dx}{X^4}\quad\text{etc.},$$

quorum primum arcu circulari solo exprimitur, reliqua vero praeterea partes algebraicas continent.

SCHOLION

73. Sufficit autem integralia $\int\frac{dx}{X^{n+1}}$ nosse, quia formula $\int\frac{(A+Bx)dx}{X^{n+1}}$ facile eo reducitur; ita enim repraesentari potest

$$\frac{1}{2bb}\int\frac{2Abbdx+2Bbbxdx-2Babdx\cos.\zeta+2Babdx\cos.\zeta}{X^{n+1}},$$

quae ob $2bbxdx-2abdx\cos.\zeta=dX$ abit in hanc

$$\frac{1}{2bb}\int\frac{BdX}{X^{n+1}}+\frac{1}{b}\int\frac{(Ab+Ba\cos.\zeta)dx}{X^{n+1}}.$$

At

$$\int\frac{dX}{X^{n+1}}=-\frac{1}{nX^n},$$

unde habebitur

$$\int\frac{(A+Bx)dx}{X^{n+1}}=\frac{-B}{2nbbX^n}+\frac{Ab+Ba\cos.\zeta}{b}\int\frac{dx}{X^{n+1}},$$

unde tantum opus est nosse integralia $\int\frac{dX}{X^{n+1}}$, quae modo exhibuimus.

Atque haec sunt omnia subsidia, quibus indigemus ad omnes formulas fractas $\frac{M}{N}dx$ integrandas, dummodo M et N sint functiones integrae ipsius x. Quocirca in genere integratio omnium huiusmodi formularum $\int Vdx$, ubi V est functio rationalis ipsius x quaecunque, est in potestate; de quibus notandum est, nisi integralia fuerint algebraica, semper vel per logarithmos vel angulos exhiberi posse. Nihil aliud igitur superest, nisi ut hanc methodum aliquot exemplis illustremus.

EXEMPLUM 1

74. *Proposita formula differentiali* $\frac{(A+Bx)dx}{\alpha+\beta x+\gamma xx}$ *definire eius integrale.*

Cum in numeratore variabilis x pauciores habeat dimensiones quam in denominatore, haec fractio nullas partes integras complectitur. Hinc denominatoris indoles perpendatur, utrum habeat duos factores simplices reales necne, ac priori casu, num factores sint aequales; ex quo tres habebimus casus evolvendos.

I. Habeat denominator ambos factores aequales sitque $=(a+bx)^2$ et fractio $\frac{A+Bx}{(a+bx)^2}$ resolvitur in has duas

$$\frac{Ab-Ba}{b(a+bx)^2}+\frac{B}{b(a+bx)},$$

unde fit

$$\int\frac{(A+Bx)dx}{(a+bx)^2}=\frac{Ba-Ab}{bb(a+bx)}+\frac{B}{bb}l(a+bx)+\text{Const.};$$

si integrale ita determinetur, ut evanescat posito $x=0$, reperitur

$$\int\frac{(A+Bx)dx}{(a+bx)^2}=\frac{(Ab-Ba)x}{ab(a+bx)}+\frac{B}{bb}l\frac{a+bx}{a}.$$

II. Habeat denominator duos factores inaequales sitque proposita haec formula $\frac{A+Bx}{(a+bx)(f+gx)}dx$ et haec fractio resolvitur in has partiales

$$\frac{Ab-Ba}{bf-ag}\cdot\frac{dx}{a+bx}+\frac{Ag-Bf}{ag-bf}\cdot\frac{dx}{f+gx},$$

unde obtinetur integrale quaesitum

$$\int\frac{(A+Bx)dx}{(a+bx)(f+gx)}=\frac{Ab-Ba}{b(bf-ag)}l\frac{a+bx}{a}+\frac{Ag-Bf}{g(ag-bf)}l\frac{f+gx}{f}+\text{Const.}$$

erit illum valorem substituendo

$$\int\frac{dx}{X^3}=\frac{-a\cos.\zeta+bx}{4aab\sin.\zeta^2X^2}+\frac{3(-a\cos.\zeta+bx)}{2\cdot 4a^4b\sin.\zeta^4X}+\frac{1\cdot 3}{2\cdot 4a^5b\sin.\zeta^5}\text{Arc. tang.}\frac{bx\sin.\zeta}{a-bx\cos.\zeta}$$

hincque porro concluditur

$$\int\frac{dx}{X^4}=\frac{-a\cos.\zeta+bx}{6aab\sin.\zeta^2X^3}+\frac{5(-a\cos.\zeta+bx)}{4\cdot 6a^4b\sin.\zeta^4X^2}+\frac{3\cdot 5(-a\cos.\zeta+bx)}{2\cdot 4\cdot 6a^6b\sin.\zeta^6X}$$
$$+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6a^7b\sin.\zeta^7}\text{Arc. tang.}\frac{bx\sin.\zeta}{a-bx\cos.\zeta}.$$

COROLLARIUM 3

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$$\int\frac{dx}{X},\quad\int\frac{dx}{X^2},\quad\int\frac{dx}{X^3},\quad\int\frac{dx}{X^4}\quad\text{etc.},$$

quorum primum arcu circulari solo exprimitur, reliqua vero praeterea partes algebraicas continent.

SCHOLION

73. Sufficit autem integralia $\int\frac{dx}{X^{n+1}}$ nosse, quia formula $\int\frac{(A+Bx)dx}{X^{n+1}}$ facile eo reducitur; ita enim repraesentari potest

$$\frac{1}{2bb}\int\frac{2Abbdx+2Bbbxdx-2Babdx\cos.\zeta+2Babdx\cos.\zeta}{X^{n+1}},$$

quae ob $2bbxdx-2abdx\cos.\zeta=dX$ abit in hanc

$$\frac{1}{2bb}\int\frac{BdX}{X^{n+1}}+\frac{1}{b}\int\frac{(Ab+Ba\cos.\zeta)dx}{X^{n+1}}.$$

At

$$\int\frac{dX}{X^{n+1}}=-\frac{1}{nX^n},$$

unde habebitur

$$\int\frac{(A+Bx)dx}{X^{n+1}}=\frac{-B}{2nbbX^n}+\frac{Ab+Ba\cos.\zeta}{b}\int\frac{dx}{X^{n+1}},$$

unde tantum opus est nosse integralia $\int\frac{dx}{X^{n+1}}$, quae modo exhibuimus.

Atque haec sunt omnia subsidia, quibus indigemus ad omnes formulas fractas $\frac{M}{N}dx$ integrandas, dummodo M et N sint functiones integrae ipsius x. Quocirca in genere integratio omnium huiusmodi formularum $\int Vdx$, ubi V est functio rationalis ipsius x quaecunque, est in potestate; de quibus notandum est, nisi integralia fuerint algebraica, semper vel per logarithmos vel angulos exhiberi posse. Nihil aliud igitur superest, nisi ut hanc methodum aliquot exemplis illustremus.

EXEMPLUM 1

74. *Proposita formula differentiali* $\frac{(A+Bx)dx}{\alpha+\beta x+\gamma xx}$ *definire eius integrale.*

Cum in numeratore variabilis x pauciores habeat dimensiones quam in denominatore, haec fractio nullas partes integras complectitur. Hinc denominatoris indoles perpendatur, utrum habeat duos factores simplices reales necne, ac priori casu, num factores sint aequales; ex quo tres habebimus casus evolvendos.

I. Habeat denominator ambos factores aequales sitque $=(a+bx)^2$ et fractio $\frac{A+Bx}{(a+bx)^2}$ resolvitur in has duas

$$\frac{Ab-Ba}{b(a+bx)^2}+\frac{B}{b(a+bx)},$$

unde fit

$$\int\frac{(A+Bx)dx}{(a+bx)^2}=\frac{Ba-Ab}{bb(a+bx)}+\frac{B}{bb}l(a+bx)+\text{Const.};$$

si integrale ita determinetur, ut evanescat posito $x=0$, reperitur

$$\int\frac{(A+Bx)dx}{(a+bx)^2}=\frac{(Ab-Ba)x}{ab(a+bx)}+\frac{B}{bb}l\frac{a+bx}{a}.$$

II. Habeat denominator duos factores inaequales sitque proposita haec formula $\frac{A+Bx}{(a+bx)(f+gx)}dx$ et haec fractio resolvitur in has partiales

$$\frac{Ab-Ba}{bf-ag}\cdot\frac{dx}{a+bx}+\frac{Ag-Bf}{ag-bf}\cdot\frac{dx}{f+gx},$$

unde obtinetur integrale quaesitum

$$\int\frac{(A+Bx)dx}{(a+bx)(f+gx)}=\frac{Ab-Ba}{b(bf-ag)}l\frac{a+bx}{a}+\frac{Ag-Bf}{g(ag-bf)}l\frac{f+gx}{f}+\text{Const.}$$

Ponatur

$$\frac{Ab - Ba}{b(bf - ag)} = m + n \quad \text{et} \quad \frac{Bf - Ag}{g(bf - ag)} = m - n,$$

ut integrale fiat

$$ml\frac{(a + bx)(f + gx)}{af} + nl\frac{f(a + bx)}{a(f + gx)};$$

erit

$$2m = \frac{B(bf - ag)}{bg(bf - ag)} = \frac{B}{bg} \quad \text{et} \quad 2n = \frac{2Abg - Bag - Bbf}{bg(bf - ag)};$$

erit ergo

$$\int \frac{(A + Bx)dx}{(a + bx)(f + gx)} = \frac{B}{2bg}l\frac{(a + bx)(f + gx)}{af} + \frac{2Abg - B(ag + bf)}{2bg(bf - ag)}l\frac{f(a + bx)}{a(f + gx)}.$$

III. Sint denominatoris factores simplices ambo imaginarii, quo casu formam habebit $aa - 2abx\cos.\zeta + bbxx$; qui casus cum supra [§ 64] iam sit tractatus, erit

$$\int \frac{(A + Bx)dx}{aa - 2abx\cos.\zeta + bbxx}$$

$$= \frac{B}{bb}l\frac{\sqrt{(aa - 2abx\cos.\zeta + bbxx)}}{a} + \frac{Ab + Ba\cos.\zeta}{abb\sin.\zeta}\,\text{Arc. tang.}\frac{bx\sin.\zeta}{a - bx\cos.\zeta}.$$

COROLLARIUM 1

75. Casu secundo, quo $f = a$ et $g = -b$, erit

$$\int \frac{(A + Bx)dx}{aa - bbxx} = \frac{-B}{2bb}l\frac{aa - bbxx}{aa} + \frac{A}{2ab}l\frac{a + bx}{a - bx};$$

hinc seorsim sequitur

$$\int \frac{Adx}{aa - bbxx} = \frac{A}{2ab}l\frac{a + bx}{a - bx} + C$$

et

$$\int \frac{Bxdx}{aa - bbxx} = \frac{-B}{2bb}l\frac{aa - bbxx}{aa} = \frac{B}{bb}l\frac{a}{\sqrt{(aa - bbxx)}} + C.$$

COROLLARIUM 2

76. Casu tertio, si ponamus $\cos.\zeta = 0$, habemus

$$\int \frac{(A+Bx)dx}{aa + bbxx} = \frac{B}{bb}l\frac{\sqrt{(aa + bbxx)}}{a} + \frac{A}{ab}\,\text{Arc. tang.}\frac{bx}{a} + C$$

hincque singillatim

$$\int \frac{A\,dx}{aa+bbxx} = \frac{A}{ab}\,\text{Arc. tang.}\,\frac{bx}{a} + C$$

et

$$\int \frac{Bx\,dx}{aa+bbxx} = \frac{B}{bb}\,l\,\frac{\sqrt{(aa+bbxx)}}{a} + C.$$

EXEMPLUM 2

77. *Proposita formula differentiali* $\frac{x^{m-1}dx}{1+x^n}$, *siquidem exponens* $m-1$ *minor sit quam* n, *integrale definire.*

In capite ultimo *Institutionum Calculi Differentialis*[1]) invenimus fractiones simplices, in quas haec fractio $\frac{x^{m-1}}{1+x^n}$ resolvitur, sumto π pro mensura duorum angulorum rectorum in hac forma generali contineri

$$\frac{2\sin.\frac{(2k-1)\pi}{n}\sin.\frac{m(2k-1)\pi}{n} - 2\cos.\frac{m(2k-1)\pi}{n}\left(x-\cos.\frac{(2k-1)\pi}{n}\right)}{n\left(1-2x\cos.\frac{(2k-1)\pi}{n}+xx\right)},$$

ubi pro k successive omnes numeros 1, 2, 3 etc. substitui convenit, quoad $2k-1$ numerum n superare incipiat. Hac ergo forma in dx ducta et cum generali nostra

$$\frac{(A+Bx)\,dx}{aa-2abx\cos.\zeta+bbxx}$$

comparata fit

$$a=1,\quad b=1,\quad \zeta=\frac{(2k-1)\pi}{n}$$

et

$$A=\frac{2}{n}\sin.\frac{(2k-1)\pi}{n}\sin.\frac{m(2k-1)\pi}{n}+\frac{2}{n}\cos.\frac{(2k-1)\pi}{n}\cos.\frac{m(2k-1)\pi}{n}$$

seu $A=\frac{2}{n}\cos.\frac{(m-1)(2k-1)\pi}{n}$ et

$$B=-\frac{2}{n}\cos.\frac{m(2k-1)\pi}{n},$$

1) L. Euleri *Institutiones calculi differentialis cum eius usu in analysi finitorum ac doctrina serierum,* Petropoli 1755; *Leonhardi Euleri Opera omnia,* series I, vol. 10; vide partis posterioris § 417, exemplum 1. L. S.

unde fit

$$Ab + Ba\cos.\zeta = \frac{2}{n}\sin.\frac{(2k-1)\pi}{n}\sin.\frac{m(2k-1)\pi}{n};$$

ac propterea huius partis integrale erit

$$-\frac{2}{n}\cos.\frac{m(2k-1)\pi}{n}\, l\sqrt{\left(1 - 2x\cos.\frac{(2k-1)\pi}{n} + xx\right)}$$

$$+\frac{2}{n}\sin.\frac{m(2k-1)\pi}{n}\ \text{Arc. tang.}\frac{x\sin.\frac{(2k-1)\pi}{n}}{1 - x\cos.\frac{(2k-1)\pi}{n}}.$$

Ac si n sit numerus impar, praeterea accedit fractio $\frac{\pm dx}{n(1+x)}$, cuius integrale est $\pm\frac{1}{n}l(1+x)$, ubi signum superius valet, si m impar, inferius vero, si m par. Quocirca integrale quaesitum $\int\frac{x^{m-1}dx}{1+x^n}$ sequenti modo exprimetur

$$-\frac{2}{n}\cos.\frac{m\pi}{n}\, l\sqrt{\left(1 - 2x\cos.\frac{\pi}{n} + xx\right)} + \frac{2}{n}\sin.\frac{m\pi}{n}\ \text{Arc. tang.}\frac{x\sin.\frac{\pi}{n}}{1 - x\cos.\frac{\pi}{n}}$$

$$-\frac{2}{n}\cos.\frac{3m\pi}{n}\, l\sqrt{\left(1 - 2x\cos.\frac{3\pi}{n} + xx\right)} + \frac{2}{n}\sin.\frac{3m\pi}{n}\ \text{Arc. tang.}\frac{x\sin.\frac{3\pi}{n}}{1 - x\cos.\frac{3\pi}{n}}$$

$$-\frac{2}{n}\cos.\frac{5m\pi}{n}\, l\sqrt{\left(1 - 2x\cos.\frac{5\pi}{n} + xx\right)} + \frac{2}{n}\sin.\frac{5m\pi}{n}\ \text{Arc. tang.}\frac{x\sin.\frac{5\pi}{n}}{1 - x\cos.\frac{5\pi}{n}}$$

$$-\frac{2}{n}\cos.\frac{7m\pi}{n}\, l\sqrt{\left(1 - 2x\cos.\frac{7\pi}{n} + xx\right)} + \frac{2}{n}\sin.\frac{7m\pi}{n}\ \text{Arc. tang.}\frac{x\sin.\frac{7\pi}{n}}{1 - x\cos.\frac{7\pi}{n}}$$

etc.

secundum numeros impares ipso n minores; sicque totum obtinetur integrale, si n fuerit numerus par, sin autem n sit numerus impar, insuper accedit haec pars $\pm\frac{1}{n}l(1+x)$, prout m sit numerus vel impar vel par; unde si $m = 1$, accedit insuper $+\frac{1}{n}l(1+x)$.

COROLLARIUM 1

78. Sumamus $m=1$, ut habeatur forma $\int \frac{dx}{1+x^n}$, et pro variis casibus ipsius n adipiscimur

$$\text{I. } \int \frac{dx}{1+x} = l(1+x)$$

$$\text{II. } \int \frac{dx}{1+x^2} = \text{Arc. tang. } x$$

$$\text{III. } \int \frac{dx}{1+x^3} = -\frac{2}{3}\cos.\frac{\pi}{3}\, l\sqrt{\left(1-2x\cos.\frac{\pi}{3}+xx\right)}$$
$$+\frac{2}{3}\sin.\frac{\pi}{3}\text{Arc. tang.}\frac{x\sin.\frac{\pi}{3}}{1-x\cos.\frac{\pi}{3}}+\frac{1}{3}l(1+x)$$

$$\text{IV. } \int \frac{dx}{1+x^4} = \begin{cases} -\frac{2}{4}\cos.\frac{\pi}{4}\, l\sqrt{\left(1-2x\cos.\frac{\pi}{4}+xx\right)} \\ \qquad +\frac{2}{4}\sin.\frac{\pi}{4}\text{Arc. tang.}\frac{x\sin.\frac{\pi}{4}}{1-x\cos.\frac{\pi}{4}} \\ -\frac{2}{4}\cos.\frac{3\pi}{4}\, l\sqrt{\left(1-2x\cos.\frac{3\pi}{4}+xx\right)} \\ \qquad +\frac{2}{4}\sin.\frac{3\pi}{4}\text{Arc. tang.}\frac{x\sin.\frac{3\pi}{4}}{1-x\cos.\frac{3\pi}{4}} \end{cases}$$

$$\text{V. } \int \frac{dx}{1+x^5} = \begin{cases} -\frac{2}{5}\cos.\frac{\pi}{5}\, l\sqrt{\left(1-2x\cos.\frac{\pi}{5}+xx\right)} \\ \qquad +\frac{2}{5}\sin.\frac{\pi}{5}\text{Arc. tang.}\frac{x\sin.\frac{\pi}{5}}{1-x\cos.\frac{\pi}{5}} \\ -\frac{2}{5}\cos.\frac{3\pi}{5}\, l\sqrt{\left(1-2x\cos.\frac{3\pi}{5}+xx\right)} \\ \qquad +\frac{2}{5}\sin.\frac{3\pi}{5}\text{Arc. tang.}\frac{x\sin.\frac{3\pi}{5}}{1-x\cos.\frac{3\pi}{5}}+\frac{1}{5}l(1+x) \end{cases}$$

$$\text{VI.}\quad \int \frac{dx}{1+x^6} = \begin{cases} -\frac{2}{6}\cos.\frac{\pi}{6}\, l\sqrt{\left(1-2x\cos.\frac{\pi}{6}+xx\right)} \\ \qquad +\frac{2}{6}\sin.\frac{\pi}{6}\,\text{Arc. tang.}\frac{x\sin.\frac{\pi}{6}}{1-x\cos.\frac{\pi}{6}} \\ -\frac{2}{6}\cos.\frac{3\pi}{6}\, l\sqrt{\left(1-2x\cos.\frac{3\pi}{6}+xx\right)} \\ \qquad +\frac{2}{6}\sin.\frac{3\pi}{6}\,\text{Arc. tang.}\frac{x\sin.\frac{3\pi}{6}}{1-x\cos.\frac{3\pi}{6}} \\ -\frac{2}{6}\cos.\frac{5\pi}{6}\, l\sqrt{\left(1-2x\cos.\frac{5\pi}{6}+xx\right)} \\ \qquad +\frac{2}{6}\sin.\frac{5\pi}{6}\,\text{Arc. tang.}\frac{x\sin.\frac{5\pi}{6}}{1-x\cos.\frac{5\pi}{6}} \end{cases}$$

COROLLARIUM 2

79. Loco sinuum et cosinuum valores, ubi commode fieri potest, substituendo obtinemus

$$\int \frac{dx}{1+x^3} = -\frac{1}{3}\, l\sqrt{(1-x+xx)} + \frac{1}{\sqrt{3}}\,\text{Arc. tang.}\frac{x\sqrt{3}}{2-x} + \frac{1}{3}\, l(1+x)$$

seu

$$\int \frac{dx}{1+x^3} = \frac{1}{3}\, l\frac{1+x}{\sqrt{(1-x+xx)}} + \frac{1}{\sqrt{3}}\,\text{Arc. tang.}\frac{x\sqrt{3}}{2-x}.$$

Deinde ob $\sin.\frac{\pi}{4} = \cos.\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \sin.\frac{3\pi}{4} = -\cos.\frac{3\pi}{4}$ fit

$$\int \frac{dx}{1+x^4} = +\frac{1}{2\sqrt{2}}\, l\frac{\sqrt{(1+x\sqrt{2}+xx)}}{\sqrt{(1-x\sqrt{2}+xx)}} + \frac{1}{2\sqrt{2}}\,\text{Arc. tang.}\frac{x\sqrt{2}}{1-xx},$$

tum vero

$$\int \frac{dx}{1+x^6} = \frac{1}{2\sqrt{3}}\, l\frac{\sqrt{(1+x\sqrt{3}+xx)}}{\sqrt{(1-x\sqrt{3}+xx)}} + \frac{1}{6}\,\text{Arc. tang.}\frac{3x(1-xx)}{1-4xx+x^4}.$$

EXEMPLUM 3

80. *Proposita formula differentiali* $\frac{x^{m-1}dx}{1-x^n}$, *siquidem exponens* $m-1$ *sit minor quam* n, *eius integrale definire.*

Functionis fractae $\frac{x^{m-1}}{1-x^n}$ pars ex factore quocunque oriunda hac forma continetur

$$\frac{2\sin.\frac{2k\pi}{n}\sin.\frac{2mk\pi}{n}-2\cos.\frac{2mk\pi}{n}\left(x-\cos.\frac{2k\pi}{n}\right)}{n\left(1-2x\cos.\frac{2k\pi}{n}+xx\right)},$$

quae cum forma nostra $\frac{A+Bx}{aa-2abx\cos.\zeta+bbxx}$ comparata dat

$$a=1,\quad b=1,\quad \zeta=\frac{2k\pi}{n},$$

$$A=\frac{2}{n}\sin.\frac{2k\pi}{n}\sin.\frac{2mk\pi}{n}+\frac{2}{n}\cos.\frac{2k\pi}{n}\cos.\frac{2mk\pi}{n},\quad B=-\frac{2}{n}\cos.\frac{2mk\pi}{n}$$

hincque

$$Ab+Ba\cos.\zeta=\frac{2}{n}\sin.\frac{2k\pi}{n}\sin.\frac{2mk\pi}{n}.$$

Ex quo integrale hinc oriundum erit

$$-\frac{2}{n}\cos.\frac{2km\pi}{n}\,l\sqrt{\left(1-2x\cos.\frac{2k\pi}{n}+xx\right)}+\frac{2}{n}\sin.\frac{2km\pi}{n}\,\mathrm{Arc.tang.}\frac{x\sin.\frac{2k\pi}{n}}{1-x\cos.\frac{2k\pi}{n}},$$

ubi pro k successive omnes numeri 1, 2, 3 etc. substitui debent, quamdiu $2k$ minor est quam n. Accedunt insuper hae ex fractione $\frac{1}{n(1-x)}$ et, si n est numerus par, ex fractione $\frac{\mp 1}{n(1+x)}$ oriundae integralis partes $-\frac{1}{n}l(1-x)$ et $\pm\frac{1}{n}l(1+x)$, ubi signum superius valet, si m est par, inferius vero, si m impar.[1])

1) Editio princeps: *ubi pro k successive omnes numeri* 0, 1, 2, 3 *etc. substitui debent, quamdiu* $2k$ *non superat* n. *At casu* $k=0$ *fit integralis pars* $-\frac{1}{n}l(1-x)$: *et quando n est numerus par, ultima pars oritur ex* $2k=n$, *quae ergo erit*

$$-\frac{2}{n}\cos.m\pi\,l\sqrt{(1+2x+xx)}=-\frac{\cos.m\pi}{n}l(1+x)$$

ergo si m est par erit $\cos.m\pi=+1$, *at si m impar, fit* $\cos.m\pi=-1$. *Quocirca* . . .
Correxit L. S.

Quocirca integrale $\int \frac{x^{m-1}dx}{1-x^n}$ hoc modo exprimitur

$$-\frac{1}{n}l(1-x)$$

$$-\frac{2}{n}\cos.\frac{2m\pi}{n}\,l\sqrt{\left(1-2x\cos.\frac{2\pi}{n}+xx\right)}+\frac{2}{n}\sin.\frac{2m\pi}{n}\,\text{Arc. tang.}\frac{x\sin.\frac{2\pi}{n}}{1-x\cos.\frac{2\pi}{n}}$$

$$-\frac{2}{n}\cos.\frac{4m\pi}{n}\,l\sqrt{\left(1-2x\cos.\frac{4\pi}{n}+xx\right)}+\frac{2}{n}\sin.\frac{4m\pi}{n}\,\text{Arc. tang.}\frac{x\sin.\frac{4\pi}{n}}{1-x\cos.\frac{4\pi}{n}}$$

$$-\frac{2}{n}\cos.\frac{6m\pi}{n}\,l\sqrt{\left(1-2x\cos.\frac{6\pi}{n}+xx\right)}+\frac{2}{n}\sin.\frac{6m\pi}{n}\,\text{Arc. tang.}\frac{x\sin.\frac{6\pi}{n}}{1-x\cos.\frac{6\pi}{n}}$$

etc.

COROLLARIUM

81. Sit $m=1$ et pro n successive numeri 1, 2, 3 etc. substituantur, ut nanciscamur sequentes integrationes

I. $\int \frac{dx}{1-x}=-l(1-x)$

II. $\int \frac{dx}{1-xx}=-\frac{1}{2}l(1-x)+\frac{1}{2}l(1+x)=\frac{1}{2}l\frac{1+x}{1-x}$

III. $\int \frac{dx}{1-x^3}=-\frac{1}{3}l(1-x)-\frac{2}{3}\cos.\frac{2\pi}{3}\,l\sqrt{\left(1-2x\cos.\frac{2\pi}{3}+xx\right)}$

$$+\frac{2}{3}\sin.\frac{2\pi}{3}\,\text{Arc. tang.}\frac{x\sin.\frac{2\pi}{3}}{1-x\cos.\frac{2\pi}{3}}$$

IV. $\int \frac{dx}{1-x^4}=-\frac{1}{4}l(1-x)-\frac{2}{4}\cos.\frac{2\pi}{4}\,l\sqrt{\left(1-2x\cos.\frac{2\pi}{4}+xx\right)}$

$$+\frac{2}{4}\sin.\frac{2\pi}{4}\,\text{Arc. tang.}\frac{x\sin.\frac{2\pi}{4}}{1-x\cos.\frac{2\pi}{4}}+\frac{1}{4}l(1+x)$$

$$\text{V.}\ \int\frac{dx}{1-x^5}=\left\{\begin{array}{l}-\frac{1}{5}l(1-x)-\frac{2}{5}\cos.\frac{2\pi}{5}\,l\sqrt{\left(1-2x\cos.\frac{2\pi}{5}+xx\right)}\\ \quad+\frac{2}{5}\sin.\frac{2\pi}{5}\,\text{Arc. tang.}\,\dfrac{x\sin.\frac{2\pi}{5}}{1-x\cos.\frac{2\pi}{5}}\\ \quad-\frac{2}{5}\cos.\frac{4\pi}{5}\,l\sqrt{\left(1-2x\cos.\frac{4\pi}{5}+xx\right)}\\ \quad+\frac{2}{5}\sin.\frac{4\pi}{5}\,\text{Arc. tang.}\,\dfrac{x\sin.\frac{4\pi}{5}}{1-x\cos.\frac{4\pi}{5}}\end{array}\right.$$

$$\text{VI.}\ \int\frac{dx}{1-x^6}=\left\{\begin{array}{l}-\frac{1}{6}l(1-x)-\frac{2}{6}\cos.\frac{2\pi}{6}\,l\sqrt{\left(1-2x\cos.\frac{2\pi}{6}+xx\right)}\\ \quad+\frac{2}{6}\sin.\frac{2\pi}{6}\,\text{Arc. tang.}\,\dfrac{x\sin.\frac{2\pi}{6}}{1-x\cos.\frac{2\pi}{6}}\\ \quad-\frac{2}{6}\cos.\frac{4\pi}{6}\,l\sqrt{\left(1-2x\cos.\frac{4\pi}{6}+xx\right)}\\ \quad+\frac{2}{6}\sin.\frac{4\pi}{6}\,\text{Arc. tang.}\,\dfrac{x\sin.\frac{4\pi}{6}}{1-x\cos.\frac{4\pi}{6}}+\frac{1}{6}l(1+x)\end{array}\right.$$

EXEMPLUM 4

82. *Proposita formula differentiali* $\frac{(x^{m-1}+x^{n-m-1})dx}{1+x^n}$ *existente* $n>m-1$ *eius integrale definire.*

Ex exemplo 2 patet integralis partem quamcunque in genere esse, sumto i pro numero quocunque impare non maiore quam n,

$$-\frac{2}{n}\cos.\frac{im\pi}{n}\,l\sqrt{\left(1-2x\cos.\frac{i\pi}{n}+xx\right)}+\frac{2}{n}\sin.\frac{im\pi}{n}\,\text{Arc. tang.}\,\frac{x\sin.\frac{i\pi}{n}}{1-x\cos.\frac{i\pi}{n}},$$

$$-\frac{2}{n}\cos.\frac{i(n-m)\pi}{n}\,l\sqrt{\left(1-2x\cos.\frac{i\pi}{n}+xx\right)}+\frac{2}{n}\sin.\frac{i(n-m)\pi}{n}\,\text{Arc. tang.}\,\frac{x\sin.\frac{i\pi}{n}}{1-x\cos.\frac{i\pi}{n}}.$$

Verum est

$$\cos.\frac{i(n-m)\pi}{n} = \cos.\left(i\pi - \frac{im\pi}{n}\right) = -\cos.\frac{im\pi}{n}$$

et

$$\sin.\frac{i(n-m)\pi}{n} = \sin.\left(i\pi - \frac{im\pi}{n}\right) = +\sin.\frac{im\pi}{n},$$

unde partes logarithmicae se destruent, eritque pars integralis in genere

$$+\frac{4}{n}\sin.\frac{im\pi}{n}\,\text{Arc. tang.}\frac{x\sin.\frac{i\pi}{n}}{1-x\cos.\frac{i\pi}{n}}.$$

Ponatur commoditatis ergo angulus $\frac{\pi}{n}=\omega$ eritque

$$\begin{aligned}\int\frac{(x^{m-1}+x^{n-m-1})dx}{1+x^n} = &+\frac{4}{n}\sin.\ m\omega\ \text{Arc. tang.}\frac{x\sin.\omega}{1-x\cos.\omega}\\ &+\frac{4}{n}\sin.3m\omega\ \text{Arc. tang.}\frac{x\sin.3\omega}{1-x\cos.3\omega}\\ &+\frac{4}{n}\sin.5m\omega\ \text{Arc. tang.}\frac{x\sin.5\omega}{1-x\cos.5\omega}\\ &\quad\vdots\\ &+\frac{4}{n}\sin.\ im\omega\ \text{Arc. tang.}\frac{x\sin.i\omega}{1-x\cos.i\omega}\end{aligned}$$

sumto pro i maximo numero impare exponentem n non excedente. Si ipse numerus n sit impar, pars ex positione $i=n$ oriunda ob $\sin.m\pi=0$ evanescet. Notetur ergo hic totum integrale per meros angulos exprimi.

COROLLARIUM

83. Simili modo sequens integrale elicitur, ubi soli logarithmi relinquentur, manente $\frac{\pi}{n}=\omega$:

$$\int \frac{(x^{m-1} - x^{n-m-1})dx}{1+x^n} = -\frac{4}{n}\cos.\ m\omega\, l\, \sqrt{(1-2x\cos.\ \omega + xx)}$$
$$-\frac{4}{n}\cos.3m\omega\, l\, \sqrt{(1-2x\cos.3\omega + xx)}$$
$$-\frac{4}{n}\cos.5m\omega\, l\, \sqrt{(1-2x\cos.5\omega + xx)}$$
$$\vdots$$
$$-\frac{4}{n}\cos.\ im\omega\, l\, \sqrt{(1-2x\cos.\ i\omega + xx)},$$

donec scilicet numerus impar i non superet exponentem n.

EXEMPLUM 5

84. *Proposita formula differentiali* $\frac{(x^{m-1} - x^{n-m-1})dx}{1-x^n}$ *existente* $n > m-1$ *eius integrale definire.*

Ex exemplo 3 integralis pars quaecunque concluditur, siquidem brevitatis gratia $\frac{\pi}{n} = \omega$ statuamus,

$$-\frac{2}{n}\cos.2km\omega\, l\, \sqrt{(1-2x\cos.2k\omega + xx)} + \frac{2}{n}\sin.2km\omega \text{ Arc. tang.} \frac{x\sin.2k\omega}{1-x\cos.2k\omega}$$
$$+\frac{2}{n}\cos.2k(n-m)\omega\, l\, \sqrt{(1-2x\cos.2k\omega+xx)} - \frac{2}{n}\sin.2k(n-m)\omega \text{ Arc. tang.} \frac{x\sin.2k\omega}{1-x\cos.2k\omega}.$$

At est
$$\cos.2k(n-m)\omega = \cos.(2k\pi - 2km\omega) = \cos.2km\omega$$
et
$$\sin.2k(n-m)\omega = \sin.(2k\pi - 2km\omega) = -\sin.2km\omega,$$

unde ista pars generalis abit in
$$\frac{4}{n}\sin.2km\omega \text{ Arc. tang.} \frac{x\sin.2k\omega}{1-x\cos.2k\omega},$$

quare hinc ista integratio colligitur
$$\int \frac{(x^{m-1} - x^{n-m-1})dx}{1-x^n} = +\frac{4}{n}\sin.2m\omega \text{ Arc. tang.} \frac{x\sin.2\omega}{1-x\cos.2\omega}$$
$$+\frac{4}{n}\sin.4m\omega \text{ Arc. tang.} \frac{x\sin.4\omega}{1-x\cos.4\omega}$$
$$+\frac{4}{n}\sin.6m\omega \text{ Arc. tang.} \frac{x\sin.6\omega}{1-x\cos.6\omega}$$

numeris paribus tamdiu ascendendo, quoad exponentem n non superent.

COROLLARIUM

85. Indidem etiam haec integratio absolvitur manente $\frac{\pi}{n} = \omega$

$$\int \frac{(x^{m-1} + x^{n-m-1})dx}{1 - x^n} = -\frac{2}{n} l(1 - x)$$
$$-\frac{4}{n} \cos. 2m\omega l \sqrt{(1 - 2x \cos. 2\omega + xx)}$$
$$-\frac{4}{n} \cos. 4m\omega l \sqrt{(1 - 2x \cos. 4\omega + xx)}$$
$$-\frac{4}{n} \cos. 6m\omega l \sqrt{(1 - 2x \cos. 6\omega + xx)},$$

ubi etiam numeri pares non ultra terminum n sunt continuandi.

EXEMPLUM 6

86. *Proposita formula differentiali* $dy = \frac{dx}{x^3(1+x)(1-x^4)}$ *eius integrale invenire.*

Functio fracta per dx affecta secundum denominatoris factores est

$$\frac{1}{x^3(1+x)^2(1-x)(1+xx)},$$

quae in has fractiones simplices resolvitur

$$\frac{1}{x^3} - \frac{1}{x^2} + \frac{1}{x} - \frac{1}{4(1+x)^2} - \frac{9}{8(1+x)} + \frac{1}{8(1-x)} + \frac{1+x}{4(1+xx)} = \frac{dy}{dx},$$

unde per integrationem elicitur

$$y = -\frac{1}{2x^2} + \frac{1}{x} + lx + \frac{1}{4(1+x)} - \frac{9}{8} l(1+x) - \frac{1}{8} l(1-x)$$
$$+ \frac{1}{8} l(1+xx) + \frac{1}{4} \text{Arc. tang.}\, x,$$

quae expressio in hanc formam transmutatur

$$y = C + \frac{-2 + 2x + 5xx}{4xx(1+x)} - l\frac{1+x}{x} + \frac{1}{8} l\frac{1+xx}{1-xx} + \frac{1}{4} \text{Arc. tang.}\, x.$$

SCHOLION

87. Hoc igitur caput ita pertractare licuit, ut nihil amplius in hoc genere desiderari possit. Quoties ergo eiusmodi functio y ipsius x quaeritur, ut $\frac{dy}{dx}$ aequetur functioni rationali ipsius x, toties integratio nihil habet difficultatis, nisi forte ad denominatoris singulos factores eliciendos Algebrae praecepta non sufficiant; verum tum defectus ipsi Algebrae, non vero methodo integrandi, quam hic tractamus, est tribuendus. Deinde etiam potissimum notari convenit semper, cum $\frac{dy}{dx}$ functioni rationali ipsius x aequale ponitur, functionem y, nisi sit algebraica, alias quantitates transcendentes non involvere praeter logarithmos et angulos; ubi quidem observandum est hic perpetuo logarithmos hyperbolicos intelligi oportere, cum ipsius lx differentiale non sit $=\frac{dx}{x}$, nisi logarithmus hyperbolicus sumatur; at horum reductio ad vulgares est facillima, ita ut hinc applicatio calculi ad praxin nulli impedimento sit obnoxia. Quare progrediamur ad eos casus, quibus formula $\frac{dy}{dx}$ functioni irrationali ipsius x aequatur, ubi quidem primo notandum est, quoties ista functio per idoneam substitutionem ad rationalitatem perduci poterit, casum ad hoc caput revolvi. Veluti si fuerit

$$dy=\frac{1+\sqrt{x}-\sqrt[3]{xx}}{1+\sqrt[3]{x}}\,dx,$$

evidens est ponendo $x=z^6$, unde fit $dx=6z^5dz$, fore

$$dy=\frac{(1+z^3-z^4)}{1+zz}\cdot 6z^5dz$$

ideoque

$$\frac{dy}{dz}=-6z^7+6z^6+6z^5-6z^4+6zz-6+\frac{6}{1+zz},$$

unde integrale

$$y=-\frac{3}{4}z^8+\frac{6}{7}z^7+z^6-\frac{6}{5}z^5+2z^3-6z+6\text{ Arc. tang. }z$$

et restituto valore

$$y=-\frac{3}{4}x\sqrt[3]{x}+\frac{6}{7}x\sqrt[6]{x}+x-\frac{6}{5}\sqrt[6]{x^5}+2\sqrt{x}-6\sqrt[6]{x}+6\text{ Arc. tang. }\sqrt[6]{x}+C.$$

CAPUT II

DE INTEGRATIONE FORMULARUM DIFFERENTIALIUM IRRATIONALIUM

PROBLEMA 6

88. *Proposita formula differentiali* $dy = \frac{dx}{\sqrt{(\alpha + \beta x + \gamma xx)}}$ *eius integrale invenire.*

SOLUTIO

Quantitas $\alpha + \beta x + \gamma xx$ vel habet duos factores reales vel secus.

I. Priori casu formula proposita erit huiusmodi

$$dy = \frac{dx}{\sqrt{(a+bx)(f+gx)}};$$

statuatur ad irrationalitatem tollendam

$$(a+bx)(f+gx) = (a+bx)^2 zz;$$

erit

$$x = \frac{f-azz}{bzz-g} \quad \text{ideoque} \quad dx = \frac{2(ag-bf)zdz}{(bzz-g)^2}$$

et

$$\sqrt{(a+bx)(f+gx)} = -\frac{(ag-bf)z}{bzz-g},$$

unde fit

$$dy = \frac{-2dz}{bzz-g} = \frac{2dz}{g-bzz} \quad \text{atque} \quad z = \sqrt{\frac{f+gx}{a+bx}}.$$

Quare si litterae b et g paribus signis sunt affectae, integrale per logarithmos, sin autem signis disparibus, per angulos exprimetur.

II. Posteriori casu habebimus

$$dy = \frac{dx}{\sqrt{(aa - 2abx\cos.\zeta + bbxx)}};$$

statuatur

$$bbxx - 2abx\cos.\zeta + aa = (bx - az)^2;$$

erit

$$-2bx\cos.\zeta + a = -2bxz + azz$$

et

$$x = \frac{a(1-zz)}{2b(\cos.\zeta - z)}, \quad \text{hinc} \quad dx = \frac{adz(1 - 2z\cos.\zeta + zz)}{2b(\cos.\zeta - z)^2}$$

et

$$\sqrt{(aa - 2abx\cos.\zeta + bbxx)} = \frac{a(1 - 2z\cos.\zeta + zz)}{2(\cos.\zeta - z)},$$

ergo

$$dy = \frac{dz}{b(\cos.\zeta - z)} \quad \text{et} \quad y = -\frac{1}{b}l(\cos.\zeta - z).$$

At est

$$z = \frac{bx - \sqrt{(aa - 2abx\cos.\zeta + bbxx)}}{a}$$

ideoque

$$y = -\frac{1}{b}l\frac{a\cos.\zeta - bx + \sqrt{(aa - 2abx\cos.\zeta + bbxx)}}{a}$$

vel

$$y = \frac{1}{b}l\big(-a\cos.\zeta + bx + \sqrt{(aa - 2abx\cos.\zeta + bbxx)}\big) + C.$$

COROLLARIUM 1

89. Casus ultimus latius patet et ad formulam

$$dy = \frac{dx}{\sqrt{(\alpha + \beta x + \gamma xx)}}$$

accommodari potest, dummodo fuerit γ quantitas positiva; namque ob $b = \sqrt{\gamma}$ et $a\cos.\zeta = \frac{-\beta}{2\sqrt{\gamma}}$ oritur

$$y = \frac{1}{\sqrt{\gamma}}l\left(\frac{\beta}{2\sqrt{\gamma}} + x\sqrt{\gamma} + \sqrt{(\alpha + \beta x + \gamma xx)}\right) + C$$

seu

$$y = \frac{1}{\sqrt{\gamma}}l\left(\frac{1}{2}\beta + \gamma x + \sqrt{\gamma(\alpha + \beta x + \gamma xx)}\right) + C.$$

COROLLARIUM 2

90. Pro casu priori cum sit

$$\int\frac{2dz}{g-bzz}=\frac{1}{\sqrt{bg}}l\frac{\sqrt{g}+z\sqrt{b}}{\sqrt{g}-z\sqrt{b}}\quad\text{et}\quad\int\frac{2dz}{g+bzz}=\frac{2}{\sqrt{gb}}\text{Arc. tang.}\frac{z\sqrt{b}}{\sqrt{g}},$$

habebimus hos casus

$$\int\frac{dx}{\sqrt{(a+bx)(f+gx)}}=\frac{1}{\sqrt{bg}}l\frac{\sqrt{g(a+bx)}+\sqrt{b(f+gx)}}{\sqrt{g(a+bx)}-\sqrt{b(f+gx)}}+C$$

$$\int\frac{dx}{\sqrt{(bx-a)(f+gx)}}=\frac{1}{\sqrt{bg}}l\frac{\sqrt{g(bx-a)}+\sqrt{b(f+gx)}}{\sqrt{g(bx-a)}-\sqrt{b(f+gx)}}+C$$

$$\int\frac{dx}{\sqrt{(bx-a)(gx-f)}}=\frac{1}{\sqrt{bg}}l\frac{\sqrt{g(bx-a)}+\sqrt{b(gx-f)}}{\sqrt{g(bx-a)}-\sqrt{b(gx-f)}}+C$$

$$\int\frac{dx}{\sqrt{(a-bx)(f-gx)}}=\frac{-1}{\sqrt{bg}}l\frac{\sqrt{g(a-bx)}+\sqrt{b(f-gx)}}{\sqrt{g(a-bx)}-\sqrt{b(f-gx)}}+C$$

$$\int\frac{dx}{\sqrt{(a-bx)(f+gx)}}=\frac{2}{\sqrt{bg}}\text{Arc. tang.}\frac{\sqrt{b(f+gx)}}{\sqrt{g(a-bx)}}+C$$

$$\int\frac{dx}{\sqrt{(a-bx)(gx-f)}}=\frac{2}{\sqrt{bg}}\text{Arc. tang.}\frac{\sqrt{b(gx-f)}}{\sqrt{g(a-bx)}}+C.$$

COROLLARIUM 3

91. Harum sex integrationum quatuor priores omnes in casu Corollarii 1 continentur, binae autem postremae in hac formula

$$dy=\frac{dx}{\sqrt{(\alpha+\beta x-\gamma xx)}}$$

continentur; sit enim pro penultima

$$af=\alpha,\quad ag-bf=\beta,\quad bg=\gamma,$$

unde colligitur

$$y=\frac{1}{\sqrt{\gamma}}\text{Arc. tang.}\frac{2\sqrt{\gamma(\alpha+\beta x-\gamma xx)}}{\beta-2\gamma x},$$

si scilicet ille arcus duplicetur. Per cosinum autem erit

$$y = \frac{1}{\sqrt{\gamma}} \text{Arc. cos.} \frac{\beta - 2\gamma x}{\sqrt{(\beta\beta + 4\alpha\gamma)}} + C,$$

cuius veritas ex differentiatione patet.

SCHOLION 1

92. Ex solutione huius problematis patet etiam hanc formulam latius patentem

$$\frac{X dx}{\sqrt{(\alpha + \beta x + \gamma xx)}},$$

si X fuerit functio rationalis quaecunque ipsius x, per praecepta capitis praecedentis integrari posse. Introducta enim loco x variabili z, qua formula radicalis rationalis redditur, etiam X abibit in functionem rationalem ipsius z. Idem adhuc generalius locum habet, si posito $\sqrt{(\alpha + \beta x + \gamma xx)} = u$ fuerit X functio quaecunque rationalis binarum quantitatum x et u; tum enim per substitutionem adhibitam, quia tam pro x quam pro u formulae rationales ipsius z scribuntur, prodibit formula differentialis rationalis. Hoc idem etiam ita enunciari potest, ut dicamus formulae $X dx$, si functio X nullam aliam irrationalem praeter $\sqrt{(\alpha + \beta x + \gamma xx)}$ involvat, integrale assignari posse, propterea quod ea ope substitutionis in formulam differentialem rationalem transformari potest.

SCHOLION 2

93. Proposita autem formula differentiali quacunque irrationali ante omnia videndum est, num ea ope cuiuspiam substitutionis in rationalem transformari possit; quod si succedat, integratio per praecepta capitis praecedentis absolvi poterit, unde simul intelligitur integrale, nisi sit algebraicum, alias quantitates transcendentes non involvere praeter logarithmos et angulos.

Quodsi autem nulla substitutio ad hoc idonea inveniri possit, ab integrationis labore est desistendum, quandoquidem integrale neque algebraice neque per logarithmos vel angulos exprimere valemus. Veluti si $X dx$ fuerit eiusmodi formula differentialis, quae nullo pacto ad rationalitatem reduci queat, eius integrale $\int X dx$ ad novum genus functionum transcendentium erit referendum, in quo nihil aliud nobis relinquitur, nisi ut eius valorem vero proxime assignare conemur. Admisso autem novo genere quantitatum transcendentium innumerabiles aliae formulae eo reduci atque integrari poterunt. Imprimis igitur in hoc erit elaborandum, ut pro quolibet genere formula simplicissima notetur, qua concessa reliquarum formularum integralia definire liceat. Hinc deducimur ad quaestionem maximi momenti, quomodo integrationem formularum magis complicatarum ad simpliciores reduci oporteat. Quod antequam aggrediamur, alias eiusmodi formulas perpendamus, quae ope idoneae substitutionis ab irrationalitate liberari queant, quemadmodum iam ostendimus, quoties X fuerit functio rationalis quantitatum x et $u=\sqrt{(\alpha+\beta x+\gamma xx)}$, ita ut alia irrationalitas non ingrediatur praeter radicem quadratam huiusmodi formulae $\alpha+\beta x+\gamma xx$, toties formulam differentialem Xdx in rationalem transformari posse.

PROBLEMA 7

94. *Proposita formula differentiali* $Xdx(a+bx)^{\frac{\mu}{\nu}}$, *in qua* X *denotet functionem quamcunque rationalem ipsius* x, *eam ab irrationalitate liberare.*

SOLUTIO

Statuatur

$$a+bx=z^{\nu},$$

ut fiat

$$(a+bx)^{\frac{\mu}{\nu}}=z^{\mu};$$

tum, quia $x=\frac{z^{\nu}-a}{b}$, facta hac substitutione functio X abibit in functionem rationalem ipsius z, quae sit Z, et ob $dx=\frac{\nu}{b}z^{\nu-1}dz$ formula nostra differentialis induet hanc formam $\frac{\nu}{b}Zz^{\mu+\nu-1}dz$; quae cum sit rationalis, per caput superius integrari potest et integrale, nisi sit algebraicum, per logarithmos et angulos exprimetur.

COROLLARIUM 1

95. Hac substitutione generalius negotium confici poterit, si posito $(a+bx)^{\frac{1}{\nu}}=u$ littera V denotet functionem quamcunque rationalem binarum quantitatum x et u; cum enim posito $x=\frac{u^{\nu}-a}{b}$ fiat V functio rationalis ipsius u, formula $Vdx=\frac{\nu}{b}Vu^{\nu-1}du$ erit rationalis.

COROLLARIUM 2

96. Quin etiam si binae irrationalitates eiusdem quantitatis $a+bx$, scilicet $(a+bx)^{\frac{1}{\nu}}=u$ et $(a+bx)^{\frac{1}{n}}=v$, ingrediantur in formulam Xdx, posito $a+bx=z^{n\nu}$ fit $x=\frac{z^{n\nu}-a}{b}$, $u=z^n$ et $v=z^\nu$; unde cum X fiat functio rationalis ipsius z et $dx=\frac{n\nu}{b}z^{n\nu-1}dz$, hac substitutione formula Xdx evadet rationalis.

COROLLARIUM 3

97. Eodem modo intelligitur, si posito

$$(a+bx)^{\frac{1}{\lambda}}=u,\quad (a+bx)^{\frac{1}{\mu}}=v,\quad (a+bx)^{\frac{1}{\nu}}=t \quad \text{etc.}$$

littera X denotet functionem quamcunque rationalem quantitatum x, u, v, t etc., formulam differentialem Xdx rationalem reddi facto

$$a+bx=z^{\lambda\mu\nu};$$

erit enim

$$x=\frac{z^{\lambda\mu\nu}-a}{b},\quad u=z^{\mu\nu},\quad v=z^{\lambda\nu},\quad t=z^{\lambda\mu} \text{ etc. et } dx=\frac{\lambda\mu\nu}{b}z^{\lambda\mu\nu-1}dz.$$

EXEMPLUM

98. Proposita hac formula

$$dy=\frac{xdx}{\sqrt[3]{(1+x)}-\sqrt{(1+x)}}$$

facto $1+x=z^6$ reperitur

$$dy=-\frac{6z^3dz(1-z^6)}{1-z}$$

seu

$$dy=-6dz(z^3+z^4+z^5+z^6+z^7+z^8)$$

hincque integrando

$$y=C-\frac{3}{2}z^4-\frac{6}{5}z^5-z^6-\frac{6}{7}z^7-\frac{3}{4}z^8-\frac{2}{3}z^9$$

et restituendo

$$y = C - \frac{3}{2}\sqrt[3]{(1+x)^2} - \frac{6}{5}\sqrt[6]{(1+x)^5} - 1 - x - \frac{6}{7}(1+x)\sqrt[6]{(1+x)}$$
$$- \frac{3}{4}(1+x)\sqrt[3]{(1+x)} - \frac{2}{3}(1+x)\sqrt{(1+x)},$$

ita ut integrale adeo algebraice exhibeatur.

PROBLEMA 8

99. *Proposita formula differentiali* $Xdx\left(\frac{a+bx}{f+gx}\right)^{\frac{\mu}{\nu}}$ *denotante* X *functionem rationalem quamcunque ipsius* x *eam ab irrationalitate liberare.*

SOLUTIO

Posito

$$\frac{a+bx}{f+gx} = z^\nu$$

fit

$$\left(\frac{a+bx}{f+gx}\right)^{\frac{\mu}{\nu}} = z^\mu$$

et

$$x = \frac{a-fz^\nu}{gz^\nu - b} \quad \text{atque} \quad dx = \frac{\nu(bf-ag)z^{\nu-1}dz}{(gz^\nu - b)^2}$$

sicque loco X prodibit functio rationalis ipsius z, qua posita $= Z$ erit formula nostra differentialis

$$= \frac{\nu(bf-ag)Zz^{\mu+\nu-1}dz}{(gz^\nu - b)^2},$$

quae cum sit rationalis, per praecepta Capitis I integrari poterit.

COROLLARIUM 1

100. Posito $\left(\frac{a+bx}{f+gx}\right)^{\frac{1}{\nu}} = u$ si X fuerit functio quaecunque rationalis binarum quantitatum x et u, formula differentialis Xdx per substitutionem usurpatam in rationalem transformabitur, cuius propterea integratio constat.

COROLLARIUM 2

101. Si X fuerit functio rationalis tam ipsius x quam quantitatum quotcunque huiusmodi

$$\left(\frac{a+bx}{f+gx}\right)^{\frac{1}{\lambda}}=u, \quad \left(\frac{a+bx}{f+gx}\right)^{\frac{1}{\mu}}=v, \quad \left(\frac{a+bx}{f+gx}\right)^{\frac{1}{\nu}}=t,$$

tum formula differentialis Xdx rationalis reddetur adhibita substitutione

$$\frac{a+bx}{f+gx}=z^{\lambda\mu\nu},$$

unde fit

$$x=\frac{a-fz^{\lambda\mu\nu}}{gz^{\lambda\mu\nu}-b} \quad \text{et} \quad u=z^{\mu\nu}, \quad v=z^{\lambda\nu}, \quad t=z^{\lambda\mu}.$$

SCHOLION 1

102. His casibus reductio ad rationalitatem ideo succedit, etiamsi plures formulae irrationales insint, quod eae omnes simul per eandem substitutionem rationales efficiantur indeque etiam ipsa quantitas x per novam variabilem z rationaliter exprimetur. Sin autem differentiale propositum duas eiusmodi formulas irrationales contineat, quae non ambae simul ope eiusdem substitutionis rationales reddi queant, etiamsi hoc in utraque seorsim fieri possit, reductio locum non habet, nisi forte ipsum differentiale in duas partes dispesci liceat, quarum utraque unam tantum formulam irrationalem complectatur. Veluti si proposita sit haec formula differentialis

$$dy=\frac{dx}{\sqrt{(1+xx)}-\sqrt{(1-xx)}},$$

eius numeratorem ac denominatorem per $\sqrt{(1+xx)}+\sqrt{(1-xx)}$ multiplicando fit

$$dy=\frac{dx\sqrt{(1+xx)}}{2xx}+\frac{dx\sqrt{(1-xx)}}{2xx},$$

cuius utraque pars seorsim rationalis reddi et integrari potest. Reperitur autem

$$y=C-\frac{\sqrt{(1-xx)}+\sqrt{(1+xx)}}{2x}+\frac{1}{2}\,l\left(x+\sqrt{(1+xx)}\right)-\frac{1}{2}\,\text{Arc. tang.}\,\frac{x}{\sqrt{(1-xx)}}\ ^{1}).$$

1) Editio princeps: $y=C-\frac{\sqrt{(1-xx)}-\sqrt{(1+xx)}}{2x}+\cdots$ Correxit L. S.

Commodissime autem ibi irrationalitas tollitur, si in parte priori ponatur $\sqrt{(1+xx)} = px$, in posteriori $\sqrt{(1-xx)} = qx$. Etsi enim hinc sit

$$x = \frac{1}{\sqrt{(pp-1)}} \quad \text{et} \quad x = \frac{1}{\sqrt{(1+qq)}},$$

tamen oritur rationaliter

$$dy = \frac{-ppdp}{2(pp-1)} - \frac{qqdq}{2(1+qq)}.$$

SCHOLION 2

103. Circa formulas generales, quae ab irrationalitate liberari queant, vix quicquam amplius praecipere licet, dummodo hunc casum addamus, quo functio X binas huiusmodi formulas radicales $\sqrt{(a+bx)}$ et $\sqrt{(f+gx)}$ complectitur. Posito enim

$$(a+bx) = (f+gx)tt$$

fit

$$x = \frac{a-ftt}{gtt-b}$$

atque [ob]

$$\sqrt{(a+bx)} = \frac{t\sqrt{(ag-bf)}}{\sqrt{(gtt-b)}}, \quad \sqrt{(f+gx)} = \frac{\sqrt{(ag-bf)}}{\sqrt{(gtt-b)}}$$

[inerit] in formula differentiali unica tantum formula irrationalis $\sqrt{(gtt-b)}$, quae nova substitutione facile tolletur per ea, quae Problemate 6 tradidimus.

Ut igitur ad alia pergamus, imprimis considerari meretur haec formula differentialis

$$x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}},$$

cuius ob simplicitatem usus per universam Analysin est amplissimus, ubi quidem sumimus litteras m, n, μ, ν numeros integros denotare; nisi enim tales essent, facile ad hanc formam reducerentur. Veluti si haberemus $x^{-\frac{1}{3}}dx(a+b\sqrt{x})^{\frac{\mu}{\nu}}$, statui oportet $x = u^6$, hinc $dx = 6u^5du$, unde prodit $6u^3du(a+bu^3)^{\frac{\mu}{\nu}}$. Tum vero pro n valorem positivum assumere licet; si enim esset negativus, puta $x^{m-1}dx(a+bx^{-n})^{\frac{\mu}{\nu}}$, ponatur $x = \frac{1}{u}$ fietque formula $-u^{-m-1}du(a+bu^n)^{\frac{\mu}{\nu}}$ similis principali; quae ergo quibus casibus ab irrationalitate liberari queat, investigemus.

PROBLEMA 9

104. *Definire casus, quibus formulam differentialem* $x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}}$ *ad rationalitatem perducere liceat.*

SOLUTIO

Primo patet, si fuerit $\nu=1$ seu $\frac{\mu}{\nu}$ numerus integer, formulam per se fore rationalem neque substitutione opus esse. At si $\frac{\mu}{\nu}$ sit fractio, substitutione est utendum eaque duplici.

I. Ponatur

$$a+bx^n=u^\nu,$$

ut fiat

$$(a+bx^n)^{\frac{\mu}{\nu}}=u^\mu;$$

erit $x^n=\frac{u^\nu-a}{b}$, hinc

$$x^m=\left(\frac{u^\nu-a}{b}\right)^{\frac{m}{n}} \quad \text{ideoque} \quad x^{m-1}dx=\frac{\nu}{nb}u^{\nu-1}du\left(\frac{u^\nu-a}{b}\right)^{\frac{m-n}{n}},$$

unde formula nostra fiet

$$\frac{\nu}{nb}u^{\mu+\nu-1}du\left(\frac{u^\nu-a}{b}\right)^{\frac{m-n}{n}}.$$

Hinc ergo patet, quoties exponens $\frac{m-n}{n}$ seu $\frac{m}{n}$ fuerit numerus integer sive positivus sive negativus, hanc formulam esse rationalem.

II. Ponatur

$$a+bx^n=x^nz^\nu,$$

ut fiat

$$x^n=\frac{a}{z^\nu-b} \quad \text{et} \quad (a+bx^n)^{\frac{\mu}{\nu}}=\frac{a^{\frac{\mu}{\nu}}z^\mu}{(z^\nu-b)^{\frac{\mu}{\nu}}};$$

tum

$$x^m=\frac{a^{\frac{m}{n}}}{(z^\nu-b)^{\frac{m}{n}}}, \quad \text{hinc} \quad x^{m-1}dx=\frac{-\nu a^{\frac{m}{n}}z^{\nu-1}dz}{n(z^\nu-b)^{\frac{m}{n}+1}}$$

ideoque formula nostra erit

$$\frac{-\nu a^{\frac{m}{n}+\frac{\mu}{\nu}} z^{\mu+\nu-1}dz}{n(z^\nu-b)^{\frac{m}{n}+\frac{\mu}{\nu}+1}}.$$

Ex quo patet hanc formam fore rationalem, quoties $\frac{m}{n}+\frac{\mu}{\nu}$ fuerit numerus integer.

Facile autem intelligitur alias substitutiones huic scopo idoneas excogitari non posse. Quare concludimus formulam irrationalem hanc $x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}}$ ab irrationalitate liberari posse, si fuerit vel $\frac{m}{n}$ vel $\frac{m}{n}+\frac{\mu}{\nu}$ numerus integer.[1])

COROLLARIUM 1

105. Si sit $\frac{m}{n}$ numerus integer, casus per se est facilis; ponatur enim $m=in$ et sit $x^n=v$; erit $x^m=v^i$ ideoque formula nostra

$$\frac{i}{m}v^{i-1}dv(a+bv)^{\frac{\mu}{\nu}},$$

quae per Problema 7 expeditur.

COROLLARIUM 2

106. At si $\frac{m}{n}$ non est numerus integer, ut reductio ad rationalitatem locum habeat, necesse est, ut $\frac{m}{n}+\frac{\mu}{\nu}$ sit numerus integer, quod fieri nequit, nisi sit $\nu=n$, ideoque $m+\mu$ multiplum debet esse ipsius $n=\nu$.

COROLLARIUM 3

107. Quodsi ergo haec formula $x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}}$ ad rationalitatem reduci queat, etiam haec formula $x^{m\pm\alpha n-1}dx(a+bx^n)^{\frac{\mu}{\nu}\pm\beta}$ eandem reductionem admittet, quicunque numeri integri pro α et β assumantur. Unde ad casus reducibiles cognoscendos sufficit ponere $m<n$ et $\mu<\nu$.

1) Hanc EULERI assertionem confirmavit P. L. TCHEBYCHEF in commentatione *Sur l'intégration des différentielles irrationnelles*, Journ. de Mathém. 18, 1853, p. 106; *Oeuvres de P. L. Tchebychef, publiées par les soins de MM. A. Markoff et N. Sonin*, St.-Pétersbourg 1899—1907, t. I, p. 163. Cf. quoque, quae inseruit C. F. GAUSS in catalogum suum d. X m. Ian. a. 1797: „*Criterii Euleriani rationem sponte detexi*"; vide commentationem a F. KLEIN editam *Gauss' wissenschaftliches Tagebuch* 1796—1814, Mathem. Ann. 57, 1903, p. 12. L. S.

COROLLARIUM 4

108. Si $m=0$, haec formula $\frac{dx}{x}(a+bx^n)^{\frac{\mu}{\nu}}$ semper per casum primum ad rationalitatem reducitur ponendo $x^n=\frac{u^\nu-a}{b}$; transformatur enim in hanc $\frac{\nu u^{\mu+\nu-1}du}{n(u^\nu-a)}$.

SCHOLION 1

109. Quoniam formula $x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}}$, quoties est $m=in$ denotante i numerum integrum sive positivum sive negativum quemcunque, semper ad rationalitatem reduci potest hique casus per se sunt perspicui, reliquos casus hanc reductionem admittentes accuratius contemplari operae pretium videtur. Quem in finem statuamus $\nu=n$ et $m<n$, item $\mu<n$, ac necesse est, ut sit $m+\mu=n$; unde sequentes formae in genere suo simplicissimae, quae quidem ad rationalitatem reduci queant, obtinentur.

I. $dx(a+bx^2)^{\frac{1}{2}}$,

II. $dx(a+bx^3)^{\frac{2}{3}}$, $\quad xdx(a+bx^3)^{\frac{1}{3}}$,

III. $dx(a+bx^4)^{\frac{3}{4}}$, $\quad xxdx(a+bx^4)^{\frac{1}{4}}$,

IV. $dx(a+bx^5)^{\frac{4}{5}}$, $\quad xdx(a+bx^5)^{\frac{3}{5}}$, $\quad x^2dx(a+bx^5)^{\frac{2}{5}}$, $\quad x^3dx(a+bx^5)^{\frac{1}{5}}$,

V. $dx(a+bx^6)^{\frac{5}{6}}$, $\quad x^4dx(a+bx^6)^{\frac{1}{6}}$,

unde etiam hae reductionem admittent

$$x^{\pm 2\alpha}dx(a+bx^2)^{\frac{1}{2}\pm\beta},$$

$$x^{\pm 3\alpha}dx(a+bx^3)^{\frac{2}{3}\pm\beta}, \qquad x^{1\pm 3\alpha}dx(a+bx^3)^{\frac{1}{3}\pm\beta},$$

$$x^{\pm 4\alpha}dx(a+bx^4)^{\frac{3}{4}\pm\beta}, \qquad x^{2\pm 4\alpha}dx(a+bx^4)^{\frac{1}{4}\pm\beta},$$

$$x^{\pm 5\alpha}dx(a+bx^5)^{\frac{4}{5}\pm\beta}, \qquad x^{1\pm 5\alpha}dx(a+bx^5)^{\frac{3}{5}\pm\beta},$$

$$x^{2\pm 5\alpha}dx(a+bx^5)^{\frac{2}{5}\pm\beta},$$

$$x^{3\pm 5\alpha}dx(a+bx^5)^{\frac{1}{5}\pm\beta},$$

$$x^{\pm 6\alpha}dx(a+bx^6)^{\frac{5}{6}\pm\beta}, \qquad x^{4\pm 6\alpha}dx(a+bx^6)^{\frac{1}{6}\pm\beta}.$$

SCHOLION 2

110. Verum etiamsi formula $x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}}$ ab irrationalitate liberari nequeat, tamen semper omnium harum formularum $x^{m\pm n\alpha-1}dx(a+bx^n)^{\frac{\mu}{\nu}\pm\beta}$ integrationem ad eam reducere licet, ita ut illius integrali tanquam cognito spectato etiam harum integralia assignari queant. Quae reductio cum in Analysi summam afferat utilitatem, eam hic exponere necesse erit. Caeterum hic affirmare haud dubitamus praeter eos casus, quos reductionem ad rationalitatem admittere hic ostendimus, nullos alios existere, qui ulla substitutione adhibita ab irrationalitate liberari queant. Proposita enim hac formula $\frac{dx}{\sqrt{(a+bx^3)}}$ nulla functio rationalis ipsius z loco x poni potest, ut $a+bx^3$ extractionem radicis quadratae admittat; obiici quidem potest scopo satisfieri posse, etiamsi loco x functio irrationalis ipsius z substituatur, dummodo similis irrationalitas in denominatore $\sqrt{(a+bx^3)}$ contineatur, qua illa numeratorem dx afficiens destruatur, quemadmodum fit in hac formula $\frac{dx}{\sqrt[3]{(a+bx^3)}}$ adhibendo substitutionem $x=\frac{\sqrt[3]{a}}{\sqrt[3]{(z^3-b)}}$; verum quod hic commode usu venit, nullo modo perspicitur, quomodo idem illo casu evenire possit. Hoc tamen minime pro demonstratione haberi volo.

PROBLEMA 10

111. *Integrationem formulae* $\int x^{m+n-1}dx(a+bx^n)^{\frac{\mu}{\nu}}$ *perducere ad integrationem huius formulae* $\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}}$.

SOLUTIO

Consideretur functio $x^m(a+bx^n)^{\frac{\mu}{\nu}+1}$; cuius differentiale cum sit

$$\left(max^{m-1}dx+mbx^{m+n-1}dx+\frac{n(\mu+\nu)}{\nu}bx^{m+n-1}dx\right)(a+bx^n)^{\frac{\mu}{\nu}},$$

erit

$$x^m(a+bx^n)^{\frac{\mu}{\nu}+1}=ma\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}}+\frac{(m\nu+n\mu+n\nu)b}{\nu}\int x^{m+n-1}dx(a+bx^n)^{\frac{\mu}{\nu}},$$

unde elicitur

$$\int x^{m+n-1}dx(a+bx^n)^{\frac{\mu}{\nu}} = \frac{\nu x^m(a+bx^n)^{\frac{\mu}{\nu}+1}}{(m\nu+n\mu+n\nu)b} - \frac{m\nu a}{(m\nu+n\mu+n\nu)b}\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}}.$$

COROLLARIUM 1

112. Cum inde quoque sit

$$\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}} = \frac{x^m(a+bx^n)^{\frac{\mu}{\nu}+1}}{ma} - \frac{(m\nu+n\mu+n\nu)b}{m\nu a}\int x^{m+n-1}dx(a+bx^n)^{\frac{\mu}{\nu}},$$

loco m scribamus $m-n$ et habebimus hanc reductionem

$$\int x^{m-n-1}dx(a+bx^n)^{\frac{\mu}{\nu}} = \frac{x^{m-n}(a+bx^n)^{\frac{\mu}{\nu}+1}}{(m-n)a} - \frac{(m\nu+n\mu)b}{(m-n)\nu a}\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}}.$$

COROLLARIUM 2

113. Concesso ergo integrali $\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}}$ etiam harum formularum $\int x^{m\pm n-1}dx(a+bx^n)^{\frac{\mu}{\nu}}$ similique modo ulterius progrediendo omnium harum formularum $\int x^{m\pm \alpha n-1}dx(a+bx^n)^{\frac{\mu}{\nu}}$ integralia exhiberi possunt.

PROBLEMA 11

114. *Integrationem formulae* $\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}+1}$ *ad integrationem huius* $\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}}$ *perducere.*

SOLUTIO

Functionis $x^m(a+bx^n)^{\frac{\mu}{\nu}+1}$ differentiale hoc modo exhiberi potest

$$\left(ma - \frac{(m\nu+n\mu+n\nu)a}{\nu}\right)x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}} + \frac{m\nu+n\mu+n\nu}{\nu}x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}+1},$$

unde concluditur

$$x^m(a+bx^n)^{\frac{\mu}{\nu}+1}$$

$$=-\frac{(n\mu+n\nu)a}{\nu}\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}}+\frac{m\nu+n\mu+n\nu}{\nu}\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}+1},$$

quocirca habebimus

$$\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}+1}=\frac{\nu x^m(a+bx^n)^{\frac{\mu}{\nu}+1}}{m\nu+n(\mu+\nu)}+\frac{n(\mu+\nu)a}{m\nu+n(\mu+\nu)}\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}}.$$

COROLLARIUM 1

115. Deinde ex eadem aequatione elicimus

$$\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}}=\frac{-\nu x^m(a+bx^n)^{\frac{\mu}{\nu}+1}}{n(\mu+\nu)a}+\frac{m\nu+n(\mu+\nu)}{n(\mu+\nu)a}\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}+1};$$

scribamus iam $\mu-\nu$ loco μ, ut nanciscamur hanc reductionem

$$\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}-1}=\frac{-\nu x^m(a+bx^n)^{\frac{\mu}{\nu}}}{n\mu a}+\frac{m\nu+n\mu}{n\mu a}\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}}.$$

COROLLARIUM 2

116. Concesso ergo integrali $\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}}$ etiam harum formularum $\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}\pm 1}$ et ulterius progrediendo harum $\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}\pm\beta}$ integralia exhiberi possunt denotante β numerum integrum quemcunque.

COROLLARIUM 3

117. His cum praecedentibus coniunctis ad integrationem

$$\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}}$$

omnia haec integralia

$$\int x^{m\pm\alpha n-1}dx(a+bx^n)^{\frac{\mu}{\nu}\pm\beta}$$

revocari possunt, quae ergo omnia ab eadem functione transcendente pendent.

SCHOLION 1

118. Ex formae $x^m(a+bx^n)^{\frac{\mu}{\nu}}$ differentiali ita disposito

$$mx^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}} + \frac{n\mu}{\nu}bx^{m+n-1}dx(a+bx^n)^{\frac{\mu}{\nu}-1}$$

deducimus hanc reductionem

$$\int x^{m+n-1}dx(a+bx^n)^{\frac{\mu}{\nu}-1} = \frac{\nu x^m(a+bx^n)^{\frac{\mu}{\nu}}}{n\mu b} - \frac{m\nu}{n\mu b}\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}}$$

ac praeterea hanc inversam pro m et μ scribendo $m-n$ et $\mu+\nu$

$$\int x^{m-n-1}dx(a+bx^n)^{\frac{\mu}{\nu}+1} = \frac{x^{m-n}(a+bx^n)^{\frac{\mu}{\nu}+1}}{m-n} - \frac{n(\mu+\nu)b}{\nu(m-n)}\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}}.$$

Hinc scilicet una operatione absolvitur reductio, cum superiores formulae duplicem reductionem exigant; ex quo sex reductiones sumus nacti omnino memorabiles, quas idcirco coniunctim conspectui exponamus.

I. $$\int x^{m+n-1}dx(a+bx^n)^{\frac{\mu}{\nu}} = \frac{\nu x^m(a+bx^n)^{\frac{\mu}{\nu}+1}}{(m\nu+n(\mu+\nu))b} - \frac{m\nu a}{(m\nu+n(\mu+\nu))b}\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}},$$

II. $$\int x^{m-n-1}dx(a+bx^n)^{\frac{\mu}{\nu}} = \frac{x^{m-n}(a+bx^n)^{\frac{\mu}{\nu}+1}}{(m-n)a} - \frac{(m\nu+n\mu)b}{(m-n)\nu a}\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}},$$

III. $$\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}+1} = \frac{\nu x^m(a+bx^n)^{\frac{\mu}{\nu}+1}}{m\nu+n(\mu+\nu)} + \frac{n(\mu+\nu)a}{m\nu+n(\mu+\nu)}\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}},$$

IV. $$\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}-1} = \frac{-\nu x^m(a+bx^n)^{\frac{\mu}{\nu}}}{n\mu a} + \frac{m\nu+n\mu}{n\mu a}\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}},$$

V. $$\int x^{m+n-1}dx(a+bx^n)^{\frac{\mu}{\nu}-1} = \frac{\nu x^m(a+bx^n)^{\frac{\mu}{\nu}}}{n\mu b} - \frac{m\nu}{n\mu b}\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}},$$

VI. $$\int x^{m-n-1}dx(a+bx^n)^{\frac{\mu}{\nu}+1} = \frac{x^{m-n}(a+bx^n)^{\frac{\mu}{\nu}+1}}{m-n} - \frac{n(\mu+\nu)b}{\nu(m-n)}\int x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}}.$$

SCHOLION 2

119. Circa has reductiones primo observandum est formulam priorem algebraice esse integrabilem, si coefficiens posterioris evanescat. Ita fit

$$\text{pro I., si } m=0,\quad \int x^{n-1}\,dx(a+bx^n)^{\frac{\mu}{\nu}} = \frac{\nu(a+bx^n)^{\frac{\mu}{\nu}+1}}{n(\mu+\nu)b},$$

$$\text{pro II., si } \frac{\mu}{\nu}=\frac{-m}{n},\quad \int x^{m-n-1}dx(a+bx^n)^{\frac{-m}{n}} = \frac{x^{m-n}(a+bx^n)^{\frac{-m}{n}+1}}{(m-n)a},$$

$$\text{pro IV., si } \frac{\mu}{\nu}=\frac{-m}{n},\quad \int x^{m-1}\,dx(a+bx^n)^{\frac{-m}{n}-1} = \frac{x^m(a+bx^n)^{\frac{-m}{n}}}{ma},$$

$$\text{pro V., si } m=0,\quad \int x^{n-1}\,dx(a+bx^n)^{\frac{\mu}{\nu}-1} = \frac{\nu(a+bx^n)^{\frac{\mu}{\nu}}}{n\mu b}.$$

Deinde etiam casus notari merentur, quibus coefficiens postremae formulae fit infinitus; tum enim reductio cessat et prior formula peculiare habet integrale seorsim evolvendum.

In prima hoc evenit, si $\frac{\mu+\nu}{\nu}=\frac{-m}{n}$, et formula $\int x^{m+n-1}dx(a+bx^n)^{\frac{-m}{n}-1}$ posito $a+bx^n=x^nz^n$ seu $x^n=\frac{a}{z^n-b}$ abit in $\int\frac{-z^{-m-1}dz}{z^n-b}$, cuius integrale per caput primum definiri debet.

In secunda evenit, si $m=n$, et formula $\int\frac{dx}{x}(a+bx^n)^{\frac{\mu}{\nu}}$ posito $a+bx^n=z^\nu$ seu $x^n=\frac{z^\nu-a}{b}$ abit in $\int\frac{\nu z^{\mu+\nu-1}dz}{n(z^\nu-a)}$.

In tertia evenit, si $\frac{\mu}{\nu}=\frac{-m}{n}-1$, et formula $\int x^{m-1}dx(a+bx^n)^{\frac{-m}{n}}$ posito $a+bx^n=x^nz^n$ seu $x^n=\frac{a}{z^n-b}$ abit in $\int\frac{-z^{-m+n-1}dz}{z^n-b}$ seu posito $z=\frac{1}{u}$ in $\int\frac{u^{m-1}du}{1-bu^n}$.[1])

In quarta evenit, si $\mu=0$, et formula $\int\frac{x^{m-1}dx}{a+bx^n}$ per se est rationalis.

In quinta idem evenit, si $\mu=0$.

1) Editio princeps: *abit in* $\int\frac{-z^{-m-n-1}dz}{z^n-b}$ *seu posito* $z=\frac{1}{u}$ *in*

$$\int\frac{u^{m+2n-1}du}{1-bu^n}=\frac{-u^{m+n}}{(m+n)b}-\frac{u^m}{mbb}+\frac{1}{bb}\int\frac{u^{m-1}du}{a-bu^n}.$$ Correxit L. S.

In sexta autem, si $m = n$, et formula $\int \frac{dx}{x}(a + bx^n)^{\frac{\mu}{\nu}+1}$ posito $a + bx^n = z^\nu$ abit in $\frac{\nu}{n}\int \frac{z^{\mu+2\nu-1}dz}{z^\nu - a}$.

EXEMPLUM 1

120. *Invenire integrale huius formulae* $\int \frac{x^{m-1}dx}{\sqrt{(1-xx)}}$ *pro numeris positivis exponenti m datis.*

Hic ob $a = 1$, $b = -1$, $n = 2$, $\mu = -1$, $\nu = 2$ prima reductio dat

$$\int \frac{x^{m+1}dx}{\sqrt{(1-xx)}} = \frac{-x^m\sqrt{(1-xx)}}{m+1} + \frac{m}{m+1}\int \frac{x^{m-1}dx}{\sqrt{(1-xx)}};$$

hinc, prout pro m sumantur numeri vel impares vel pares, obtinebimus:

Pro numeris imparibus

$$\int \frac{xxdx}{\sqrt{(1-xx)}} = -\frac{1}{2}x\sqrt{(1-xx)} + \frac{1}{2}\int \frac{dx}{\sqrt{(1-xx)}},$$

$$\int \frac{x^4dx}{\sqrt{(1-xx)}} = -\frac{1}{4}x^3\sqrt{(1-xx)} + \frac{3}{4}\int \frac{x^2dx}{\sqrt{(1-xx)}},$$

$$\int \frac{x^6dx}{\sqrt{(1-xx)}} = -\frac{1}{6}x^5\sqrt{(1-xx)} + \frac{5}{6}\int \frac{x^4dx}{\sqrt{(1-xx)}}$$

etc.;

pro numeris paribus

$$\int \frac{x^3dx}{\sqrt{(1-xx)}} = -\frac{1}{3}x^2\sqrt{(1-xx)} + \frac{2}{3}\int \frac{xdx}{\sqrt{(1-xx)}},$$

$$\int \frac{x^5dx}{\sqrt{(1-xx)}} = -\frac{1}{5}x^4\sqrt{(1-xx)} + \frac{4}{5}\int \frac{x^3dx}{\sqrt{(1-xx)}},$$

$$\int \frac{x^7dx}{\sqrt{(1-xx)}} = -\frac{1}{7}x^6\sqrt{(1-xx)} + \frac{6}{7}\int \frac{x^5dx}{\sqrt{(1-xx)}}$$

etc.

Cum nunc sit

$$\int \frac{dx}{\sqrt{(1-xx)}} = \text{Arc. sin. } x \quad \text{et} \quad \int \frac{xdx}{\sqrt{(1-xx)}} = -\sqrt{(1-xx)},$$

habebimus sequentia integralia:

Pro ordine priore

$$\int\frac{dx}{\sqrt{(1-xx)}}=\text{Arc. sin. } x,$$

$$\int\frac{xxdx}{\sqrt{(1-xx)}}=-\frac{1}{2}x\sqrt{(1-xx)}+\frac{1}{2}\text{Arc. sin. } x,$$

$$\int\frac{x^4dx}{\sqrt{(1-xx)}}=-\left(\frac{1}{4}x^3+\frac{1\cdot 3}{2\cdot 4}x\right)\sqrt{(1-xx)}+\frac{1\cdot 3}{2\cdot 4}\text{Arc. sin. } x,$$

$$\int\frac{x^6dx}{\sqrt{(1-xx)}}=-\left(\frac{1}{6}x^5+\frac{1\cdot 5}{4\cdot 6}x^3+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x\right)\sqrt{(1-xx)}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\text{Arc. sin } x,$$

$$\int\frac{x^8dx}{\sqrt{(1-xx)}}=-\left(\frac{1}{8}x^7+\frac{1\cdot 7}{6\cdot 8}x^5+\frac{1\cdot 5\cdot 7}{4\cdot 6\cdot 8}x^3+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}x\right)\sqrt{(1-xx)}+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\text{Arc. sin. } x;$$

pro ordine posteriore

$$\int\frac{xdx}{\sqrt{(1-xx)}}=-\sqrt{(1-xx)},$$

$$\int\frac{x^3dx}{\sqrt{(1-xx)}}=-\left(\frac{1}{3}x^2+\frac{2}{3}\right)\sqrt{(1-xx)},$$

$$\int\frac{x^5dx}{\sqrt{(1-xx)}}=-\left(\frac{1}{5}x^4+\frac{1\cdot 4}{3\cdot 5}x^2+\frac{2\cdot 4}{3\cdot 5}\right)\sqrt{(1-xx)},$$

$$\int\frac{x^7dx}{\sqrt{(1-xx)}}=-\left(\frac{1}{7}x^6+\frac{1\cdot 6}{5\cdot 7}x^4+\frac{1\cdot 4\cdot 6}{3\cdot 5\cdot 7}x^2+\frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\right)\sqrt{(1-xx)}.$$

COROLLARIUM 1

121. In genere ergo pro formula $\int\frac{x^{2i}dx}{\sqrt{(1-xx)}}$, si ponamus brevitatis gratia $\frac{1\cdot 3\cdot 5\cdots(2i-1)}{2\cdot 4\cdot 6\cdots\cdots 2i}=J$, habebimus hoc integrale

$$\int\frac{x^{2i}dx}{\sqrt{(1-xx)}}$$

$$=J\,\text{Arc. sin. } x-J\left(x+\frac{2}{3}x^3+\frac{2\cdot 4}{3\cdot 5}x^5+\frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}x^7+\cdots+\frac{2\cdot 4\cdot 6\cdots(2i-2)}{3\cdot 5\cdot 7\cdots(2i-1)}x^{2i-1}\right)\sqrt{(1-xx)}.$$

COROLLARIUM 2

122. Simili modo pro formula $\int\frac{x^{2i+1}dx}{\sqrt{(1-xx)}}$, si ponamus brevitatis ergo $\frac{2\cdot 4\cdot 6\cdots\cdots 2i}{3\cdot 5\cdot 7\cdots(2i+1)}=K$, habebimus hoc integrale

$$\int \frac{x^{2i+1}dx}{\sqrt{(1-xx)}}$$
$$= K - K\left(1 + \frac{1}{2}x^2 + \frac{1\cdot 3}{2\cdot 4}x^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 + \cdots + \frac{1\cdot 3\cdot 5\cdots(2i-1)}{2\cdot 4\cdot 6\cdots\cdots 2i}x^{2i}\right)\sqrt{(1-xx)},$$

ut integrale evanescat posito $x = 0$.

EXEMPLUM 2

123. *Invenire integrale formulae* $\int \frac{x^{m-1}dx}{\sqrt{(1-xx)}}$ *casibus, quibus pro m numeri negativi assumuntur.*

Hic utendum est secunda reductione, quae dat

$$\int \frac{x^{m-3}dx}{\sqrt{(1-xx)}} = \frac{x^{m-2}\sqrt{(1-xx)}}{m-2} + \frac{m-1}{m-2}\int \frac{x^{m-1}dx}{\sqrt{(1-xx)}},$$

unde patet, si $m = 1$, fore

$$\int \frac{dx}{xx\sqrt{(1-xx)}} = -\frac{\sqrt{(1-xx)}}{x};$$

deinde, si $m = 2$, formula $\int \frac{dx}{x\sqrt{(1-xx)}}$ facta substitutione $1 - xx = zz$ abit in $\int \frac{-dz}{1-zz}$, cuius integrale est

$$-\frac{1}{2}l\frac{1+z}{1-z} = -\frac{1}{2}l\frac{1+\sqrt{(1-xx)}}{1-\sqrt{(1-xx)}} = -l\frac{1+\sqrt{(1-xx)}}{x},$$

unde duplicem seriem integrationum elicimus

$$\int \frac{dx}{x\sqrt{(1-xx)}} = -l\frac{1+\sqrt{(1-xx)}}{x} = l\frac{1-\sqrt{(1-xx)}}{x},$$
$$\int \frac{dx}{x^3\sqrt{(1-xx)}} = -\frac{\sqrt{(1-xx)}}{2xx} + \frac{1}{2}\int \frac{dx}{x\sqrt{(1-xx)}},$$
$$\int \frac{dx}{x^5\sqrt{(1-xx)}} = -\frac{\sqrt{(1-xx)}}{4x^4} + \frac{3}{4}\int \frac{dx}{x^3\sqrt{(1-xx)}},$$
$$\int \frac{dx}{x^7\sqrt{(1-xx)}} = -\frac{\sqrt{(1-xx)}}{6x^6} + \frac{5}{6}\int \frac{dx}{x^5\sqrt{(1-xx)}}$$

etc.

$$\int\frac{dx}{xx\sqrt{(1-xx)}}=-\frac{\sqrt{(1-xx)}}{x},$$

$$\int\frac{dx}{x^4\sqrt{(1-xx)}}=-\frac{\sqrt{(1-xx)}}{3x^3}+\frac{2}{3}\int\frac{dx}{xx\sqrt{(1-xx)}},$$

$$\int\frac{dx}{x^6\sqrt{(1-xx)}}=-\frac{\sqrt{(1-xx)}}{5x^5}+\frac{4}{5}\int\frac{dx}{x^4\sqrt{(1-xx)}}$$

etc.

Hinc erit ut in binis praecedentibus corollariis

$$\int\frac{dx}{x^{2i+1}\sqrt{(1-xx)}}$$

$$=Jl\frac{1-\sqrt{(1-xx)}}{x}-J\left(\frac{1}{xx}+\frac{2}{3x^4}+\frac{2\cdot 4}{3\cdot 5x^6}+\cdots+\frac{2\cdot 4\cdots(2i-2)}{3\cdot 5\cdots(2i-1)x^{2i}}\right)\sqrt{(1-xx)},$$

$$\int\frac{dx}{x^{2i+2}\sqrt{(1-xx)}}$$

$$=C-K\left(\frac{1}{x}+\frac{1}{2x^3}+\frac{1\cdot 3}{2\cdot 4x^5}+\cdots+\frac{1\cdot 3\cdots(2i-1)}{2\cdot 4\cdots 2ix^{2i+1}}\right)\sqrt{(1-xx)}.^{1)}$$

SCHOLION 1

124. Hinc iam facile integralia formularum $\int x^{m-1}dx(1-xx)^{\frac{\mu}{2}}$ tam pro omnibus numeris m quam pro imparibus μ assignari poterunt. Reductiones autem nostrae generales ad hunc casum accommodatae sunt:

$$\text{I. } \int x^{m+1}dx(1-xx)^{\frac{\mu}{2}}=\frac{-x^m(1-xx)^{\frac{\mu}{2}+1}}{m+\mu+2}+\frac{m}{m+\mu+2}\int x^{m-1}dx(1-xx)^{\frac{\mu}{2}},$$

$$\text{II. } \int x^{m-3}dx(1-xx)^{\frac{\mu}{2}}=\frac{x^{m-2}(1-xx)^{\frac{\mu}{2}+1}}{m-2}+\frac{m+\mu}{m-2}\int x^{m-1}dx(1-xx)^{\frac{\mu}{2}},$$

$$\text{III. } \int x^{m-1}dx(1-xx)^{\frac{\mu}{2}+1}=\frac{x^m(1-xx)^{\frac{\mu}{2}+1}}{m+\mu+2}+\frac{\mu+2}{m+\mu+2}\int x^{m-1}dx(1-xx)^{\frac{\mu}{2}},$$

1) Editio princeps: $\int\frac{dx}{x^{2i}\sqrt{(1-xx)}}=C-\cdots$ Correxit L. S.

$$\text{IV. } \int x^{m-1}dx(1-xx)^{\frac{\mu}{2}-1} = \frac{-x^m(1-xx)^{\frac{\mu}{2}}}{\mu} + \frac{m+\mu}{\mu}\int x^{m-1}dx(1-xx)^{\frac{\mu}{2}},$$

$$\text{V. } \int x^{m+1}dx(1-xx)^{\frac{\mu}{2}-1} = \frac{-x^m(1-xx)^{\frac{\mu}{2}}}{\mu} + \frac{m}{\mu}\int x^{m-1}dx(1-xx)^{\frac{\mu}{2}},$$

$$\text{VI. } \int x^{m-3}dx(1-xx)^{\frac{\mu}{2}+1} = \frac{x^{m-2}(1-xx)^{\frac{\mu}{2}+1}}{m-2} + \frac{\mu+2}{m-2}\int x^{m-1}dx(1-xx)^{\frac{\mu}{2}}.$$

Posito enim $\mu = -1$ quatuor posteriores dant

$$\int x^{m-1}dx\sqrt{(1-xx)} = \frac{x^m\sqrt{(1-xx)}}{m+1} + \frac{1}{m+1}\int\frac{x^{m-1}dx}{\sqrt{(1-xx)}},$$

$$\int\frac{x^{m-1}dx}{\sqrt{(1-xx)^3}} = \frac{x^m}{\sqrt{(1-xx)}} - (m-1)\int\frac{x^{m-1}dx}{\sqrt{(1-xx)}},$$

$$\int\frac{x^{m+1}dx}{\sqrt{(1-xx)^3}} = \frac{x^m}{\sqrt{(1-xx)}} - m\int\frac{x^{m-1}dx}{\sqrt{(1-xx)}},$$

$$\int x^{m-3}dx\sqrt{(1-xx)} = \frac{x^{m-2}\sqrt{(1-xx)}}{m-2} + \frac{1}{m-2}\int\frac{x^{m-1}dx}{\sqrt{(1-xx)}},$$

unde integrationes pro casibus $\mu = 1$ et $\mu = -3$ eliciuntur indeque porro reliqui.

SCHOLION 2

125. Pro aliis formulis irrationalibus magis complicatis vix regulas dare licet, quibus ad formam simpliciorem reduci queant; et quoties eiusmodi formulae occurrant, reductio, si quam admittunt, plerumque sponte se offert. Veluti si formula fuerit huiusmodi $\int\frac{Pdx}{Q^{n+1}}$, sive n sit numerus integer sive fractus, semper ad aliam huius formae $\int\frac{Sdx}{Q^n}$, quae utique simplicior aestimatur, reduci potest. Cum enim sit

$$d.\frac{R}{Q^n} = \frac{QdR - nRdQ}{Q^{n+1}},$$

posito $\int\frac{Pdx}{Q^{n+1}} = y$ erit

$$y + \frac{R}{Q^n} = \int\frac{Pdx + QdR - nRdQ}{Q^{n+1}}.$$

Iam definiatur R ita, ut $Pdx + QdR - nRdQ$ per Q fiat divisibile, vel quia QdR iam factorem habet Q, ut fiat $Pdx - nRdQ = QTdx$, prodibitque

$$y + \frac{R}{Q^n} = \int \frac{dR + Tdx}{Q^n}$$

seu

$$\int \frac{Pdx}{Q^{n+1}} = -\frac{R}{Q^n} + \int \frac{dR + Tdx}{Q^n};$$

at semper functionem R ita definire licet, ut $Pdx - nRdQ$ factorem Q obtineat; quod etsi in genere praestari nequit, tamen rem in exemplis tentando mox perspicietur negotium semper succedere. Assumo autem hic P et Q esse functiones integras ac talis quoque semper pro R erui poterit. Si forte eveniat, ut $dR + Tdx = 0$, formula proposita algebraicum habebit integrale, quod hoc modo reperietur; contra autem haec forma ulterius reduci poterit in alias, ubi denominatoris exponens continuo unitate diminuatur; ac si n sit numerus integer, negotium tandem reducitur ad huiusmodi formam $\frac{Vdx}{Q}$, quae sine dubio est simplicissima. Quamobrem cum in hoc capite vix quicquam amplius proferri possit ad integrationem formularum irrationalium iuvandam, methodum easdem integrationes per series infinitas perficiendi exponamus.

ADDITAMENTUM

PROBLEMA

Proposita formula $dy = (x + \sqrt{(1 + xx)})^n dx$ *invenire eius integrale.*

SOLUTIO

Posito $x + \sqrt{(1 + xx)} = u$ fit $x = \frac{uu - 1}{2u}$ et $dx = \frac{du(uu + 1)}{2uu}$, unde formula nostra

$$dy = \frac{1}{2} u^{n-2} du(uu + 1)$$

ideoque eius integrale

$$y = \frac{u^{n+1}}{2(n+1)} + \frac{u^{n-1}}{2(n-1)} + \text{Const.},$$

quod ergo semper est algebraicum, nisi sit vel $n = 1$ vel $n = -1$.

COROLLARIUM 1

Patet etiam hanc formam latius patentem

$$dy = \left(x + \sqrt{(1+xx)}\right)^n X dx$$

hoc modo integrari posse, dummodo X fuerit functio rationalis ipsius x. Posito enim $x = \frac{uu-1}{2u}$ pro X prodit functio rationalis ipsius u, quae sit $= U$, hincque fit

$$dy = \tfrac{1}{2} U u^{n-2} du (uu+1),$$

quae formula vel est rationalis, si n sit numerus integer, vel ad rationalitatem facile reducitur, si n sit numerus fractus.

COROLLARIUM 2

Cum sit $\sqrt{(1+xx)} = \frac{uu+1}{2u}$, posito $\sqrt{(1+xx)} = v$ etiam haec formula

$$dy = \left(x + \sqrt{(1+xx)}\right)^n X dx$$

integrabitur, si X fuerit functio rationalis quaecunque quantitatum x et v. Facto enim $x = \frac{uu-1}{2u}$ functio X abit in functionem rationalem ipsius u, qua posita $= U$ habebitur ut ante

$$dy = \frac{1}{2} U u^{n-2} du (uu+1).$$

EXEMPLUM

Proposita sit formula $dy = \left(ax + b\sqrt{(1+xx)}\right)\left(x + \sqrt{(1+xx)}\right)^n dx$.

Posito $x = \frac{uu-1}{2u}$ fit

$$dy = \frac{a(uu-1) + b(uu+1)}{2u} \cdot \frac{1}{2} u^{n-2} du (uu+1)$$

seu

$$dy = \frac{1}{4} u^{n-3} du \left(a(u^4-1) + b(u^4 + 2uu + 1)\right),$$

cuius integrale est

$$y = \frac{a+b}{4(n+2)} u^{n+2} + \frac{b}{2n} u^n + \frac{b-a}{4(n-2)} u^{n-2} + \text{Const.},$$

quae est algebraica, nisi sit vel $n = 2$ vel $n = -2$ vel etiam $n = 0$.

CAPUT III

DE INTEGRATIONE FORMULARUM DIFFERENTIALIUM PER SERIES INFINITAS

PROBLEMA 12

126. *Si X fuerit functio rationalis fracta ipsius x, formulae differentialis $dy = Xdx$ integrale per seriem infinitam exhibere.*

SOLUTIO

Cum X sit functio rationalis fracta, eius valor semper ita evolvi potest, ut fiat

$$X = Ax^{m} + Bx^{m+n} + Cx^{m+2n} + Dx^{m+3n} + Ex^{m+4n} + \text{etc.},$$

ubi coefficientes A, B, C etc. seriem recurrentem constituent ex denominatore fractionis determinandam. Multiplicentur ergo singuli termini per dx et integrentur, quo facto integrale y per sequentem seriem exprimetur

$$y = \frac{Ax^{m+1}}{m+1} + \frac{Bx^{m+n+1}}{m+n+1} + \frac{Cx^{m+2n+1}}{m+2n+1} + \text{etc.} + \text{Const.};$$

ubi si in serie pro X occurrat huiusmodi terminus $\frac{M}{x}$, inde in integrale ingredietur terminus Mlx.

SCHOLION

127. Cum integrale $\int Xdx$, nisi sit algebraicum, per logarithmos et angulos exprimatur, hinc valores logarithmorum et angulorum per series infinitas exhiberi possunt. Cuiusmodi series cum iam in *Introductione*[1]) plures

1) L. Euleri *Introductio in analysin infinitorum*, t. I cap. VI—VIII; Leonhardi Euleri *Opera omnia*, series I, vol. 8. L. S.

sint traditae, non solum eaedem, sed etiam infinitae aliae hic per integrationem erui possunt. Hoc exemplis declarasse iuvabit, ubi potissimum eiusmodi formulas evolvemus, in quibus denominator est binomium; tum vero etiam casus aliquot denominatore trinomio vel multinomio praeditos contemplabimur. Imprimis autem eiusmodi eligemus, quibus fractio in aliam, cuius denominator est binomius, transmutari potest.

EXEMPLUM 1

128. *Formulam differentialem $\frac{dx}{a+x}$ per seriem integrare.*

Sit $y=\int\frac{dx}{a+x}$; erit $y=l(a+x)+\text{Const.}$, unde integrali ita determinato, ut evanescat posito $x=0$, erit $y=l(a+x)-la$. Iam cum sit

$$\frac{1}{a+x}=\frac{1}{a}-\frac{x}{a^2}+\frac{xx}{a^3}-\frac{x^3}{a^4}+\frac{x^4}{a^5}-\text{etc.},$$

erit eadem lege integrale definiendo

$$y=\frac{x}{a}-\frac{x^2}{2a^2}+\frac{x^3}{3a^3}-\frac{x^4}{4a^4}+\frac{x^5}{5a^5}-\text{etc.},$$

unde colligemus, uti quidem iam constat,

$$l(a+x)=la+\frac{x}{a}-\frac{x^2}{2a^2}+\frac{x^3}{3a^3}-\frac{x^4}{4a^4}+\text{etc.}$$

COROLLARIUM 1

129. Si capiamus x negativum, ut sit $dy=\frac{-dx}{a-x}$, eodem modo patebit esse

$$l(a-x)=la-\frac{x}{a}-\frac{x^2}{2a^2}-\frac{x^3}{3a^3}-\frac{x^4}{4a^4}-\text{etc.}$$

hisque combinandis

$$l(aa-xx)=2la-\frac{xx}{aa}-\frac{x^4}{2a^4}-\frac{x^6}{3a^6}-\frac{x^8}{4a^8}-\text{etc.}$$

et

$$l\frac{a+x}{a-x}=\frac{2x}{a}+\frac{2x^3}{3a^3}+\frac{2x^5}{5a^5}+\frac{2x^7}{7a^7}+\text{etc.}$$

COROLLARIUM 2

130. Hae posteriores series eruuntur per integrationem formularum

$$\frac{-2xdx}{aa-xx} = -2xdx\left(\frac{1}{aa}+\frac{xx}{a^4}+\frac{x^4}{a^6}+\text{etc.}\right)$$

et

$$\frac{2adx}{aa-xx} = 2adx\left(\frac{1}{aa}+\frac{xx}{a^4}+\frac{x^4}{a^6}+\text{etc.}\right).$$

Est autem

$$\int\frac{-2xdx}{aa-xx} = l(aa-xx)-laa \quad \text{et} \quad \int\frac{2adx}{aa-xx} = l\frac{a+x}{a-x},$$

ita ut iam his formulis per series integrandis supersedere possimus.

EXEMPLUM 2

131. *Formulam differentialem* $\frac{adx}{aa+xx}$ *per seriem integrare.*

Sit $dy=\frac{adx}{aa+xx}$, et cum sit $y=\text{Arc. tang.}\,\frac{x}{a}$, idem angulus serie infinita exprimetur. Quia enim habemus

$$\frac{a}{aa+xx} = \frac{1}{a}-\frac{xx}{a^3}+\frac{x^4}{a^5}-\frac{x^6}{a^7}+\frac{x^8}{a^9}-\text{etc.},$$

erit integrando

$$y = \text{Arc. tang.}\,\frac{x}{a} = \frac{x}{a}-\frac{x^3}{3a^3}+\frac{x^5}{5a^5}-\frac{x^7}{7a^7}+\text{etc.}$$

EXEMPLUM 3

132. *Integralia harum formularum* $\frac{dx}{1+x^3}$ *et* $\frac{xdx}{1+x^3}$ *per series exprimere.*

Cum sit

$$\frac{1}{1+x^3} = 1-x^3+x^6-x^9+x^{12}-\text{etc.},$$

erit

$$\int\frac{dx}{1+x^3} = x-\frac{1}{4}x^4+\frac{1}{7}x^7-\frac{1}{10}x^{10}+\frac{1}{13}x^{13}-\text{etc.}$$

et

$$\int\frac{xdx}{1+x^3} = \frac{1}{2}x^2-\frac{1}{5}x^5+\frac{1}{8}x^8-\frac{1}{11}x^{11}+\frac{1}{14}x^{14}-\text{etc.}$$

Verum per § 77 habemus per logarithmos et angulos

$$\int \frac{dx}{1+x^3} = \frac{1}{3} l(1+x) - \frac{2}{3}\cos.\frac{\pi}{3}\, l\sqrt{\left(1-2x\cos.\frac{\pi}{3}+xx\right)}$$
$$+\frac{2}{3}\sin.\frac{\pi}{3}\,\text{Arc. tang.}\frac{x\sin.\frac{\pi}{3}}{1-x\cos.\frac{\pi}{3}},$$

$$\int \frac{x\,dx}{1+x^3} = -\frac{1}{3} l(1+x) - \frac{2}{3}\cos.\frac{2\pi}{3}\, l\sqrt{\left(1-2x\cos.\frac{\pi}{3}+xx\right)}$$
$$+\frac{2}{3}\sin.\frac{2\pi}{3}\,\text{Arc. tang.}\frac{x\sin.\frac{\pi}{3}}{1-x\cos.\frac{\pi}{3}}.$$

At est $\cos.\frac{\pi}{3}=\frac{1}{2}$, $\cos.\frac{2\pi}{3}=-\frac{1}{2}$, $\sin.\frac{\pi}{3}=\frac{\sqrt{3}}{2}$, $\sin.\frac{2\pi}{3}=\frac{\sqrt{3}}{2}$, unde fit

$$\int \frac{dx}{1+x^3} = \frac{1}{3} l(1+x) - \frac{1}{3} l\sqrt{(1-x+xx)} + \frac{1}{\sqrt{3}}\,\text{Arc. tang.}\frac{x\sqrt{3}}{2-x},$$
$$\int \frac{x\,dx}{1+x^3} = -\frac{1}{3} l(1+x) + \frac{1}{3} l\sqrt{(1-x+xx)} + \frac{1}{\sqrt{3}}\,\text{Arc. tang.}\frac{x\sqrt{3}}{2-x}$$

integralibus ut seriebus ita sumtis, ut evanescant posito $x=0$.

COROLLARIUM 1

133. His igitur seriebus additis prodit

$$\frac{2}{\sqrt{3}}\,\text{Arc. tang.}\frac{x\sqrt{3}}{2-x} = x + \frac{1}{2}xx - \frac{1}{4}x^4 - \frac{1}{5}x^5 + \frac{1}{7}x^7 + \frac{1}{8}x^8 - \frac{1}{10}x^{10} - \frac{1}{11}x^{11} + \text{etc.},$$

subtracta autem posteriori a priori fit

$$\frac{2}{3}\, l\frac{1+x}{\sqrt{(1-x+xx)}} = x - \frac{1}{2}x^2 - \frac{1}{4}x^4 + \frac{1}{5}x^5 + \frac{1}{7}x^7 - \frac{1}{8}x^8 - \frac{1}{10}x^{10} + \frac{1}{11}x^{11} + \text{etc.},$$

cuius valor etiam est

$$= \frac{1}{3}\, l\frac{(1+x)^2}{1-x+xx} = \frac{1}{3}\, l\frac{(1+x)^3}{1+x^3}.$$

COROLLARIUM 2

134. Cum sit

$$\int \frac{xx\,dx}{1+x^3} = \frac{1}{3}l(1+x^3),$$

erit eodem modo

$$\frac{1}{3}l(1+x^3) = \frac{1}{3}x^3 - \frac{1}{6}x^6 + \frac{1}{9}x^9 - \frac{1}{12}x^{12} + \text{etc.},$$

qua serie illis adiecta omnes potestates ipsius x occurrent.

EXEMPLUM 4

135. *Integrale hoc* $y = \int \frac{(1+xx)\,dx}{1+x^4}$ *per seriem exprimere.*

Cum sit

$$\frac{1}{1+x^4} = 1 - x^4 + x^8 - x^{12} + x^{16} - \text{etc.},$$

erit

$$y = x + \frac{1}{3}x^3 - \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 + \frac{1}{11}x^{11} - \frac{1}{13}x^{13} - \frac{1}{15}x^{15} + \text{etc.}$$

Verum per § 82, ubi $m = 1$ et $n = 4$, posito $\frac{\pi}{4} = \omega$ fit integrale idem

$$y = \sin.\omega \text{ Arc.tang.} \frac{x \sin.\omega}{1 - x\cos.\omega} + \sin.3\omega \text{ Arc.tang.} \frac{x \sin.3\omega}{1 - x\cos.3\omega};$$

at ob $\frac{\pi}{4} = \omega = 45^0$ est $\sin.\omega = \frac{1}{\sqrt{2}}$, $\cos.\omega = \frac{1}{\sqrt{2}}$, $\sin.3\omega = \frac{1}{\sqrt{2}}$, $\cos.3\omega = \frac{-1}{\sqrt{2}}$;

[hinc] habebimus

$$y = \frac{1}{\sqrt{2}} \text{Arc.tang.} \frac{x}{\sqrt{2}-x} + \frac{1}{\sqrt{2}} \text{Arc.tang.} \frac{x}{\sqrt{2}+x} = \frac{1}{\sqrt{2}} \text{Arc.tang.} \frac{x\sqrt{2}}{1-xx}.$$

EXEMPLUM 5

136. *Integrale hoc* $y = \int \frac{(1+x^4)\,dx}{1+x^6}$ *per seriem exprimere.*

Cum sit

$$\frac{1}{1+x^6} = 1 - x^6 + x^{12} - x^{18} + x^{24} - \text{etc.},$$

erit

$$y = x + \frac{1}{5}x^5 - \frac{1}{7}x^7 - \frac{1}{11}x^{11} + \frac{1}{13}x^{13} + \frac{1}{17}x^{17} - \text{etc.}$$

At per § 82, ubi $m=1$, $n=6$ et $\omega=\frac{\pi}{6}=30^0$, est

$$y=\frac{2}{3}\sin.\omega \,\text{Arc. tang.}\frac{x\sin.\omega}{1-x\cos.\omega}+\frac{2}{3}\sin.3\omega\,\text{Arc. tang.}\frac{x\sin.3\omega}{1-x\cos.3\omega}$$
$$+\frac{2}{3}\sin.5\omega\,\text{Arc. tang.}\frac{x\sin.5\omega}{1-x\cos.5\omega};$$

est vero $\sin.\omega=\frac{1}{2}$, $\cos.\omega=\frac{\sqrt{3}}{2}$, $\sin.3\omega=1$, $\cos.3\omega=0$, $\sin.5\omega=\frac{1}{2}$, $\cos.5\omega=-\frac{\sqrt{3}}{2}$, ergo

$$y=\frac{1}{3}\text{Arc. tang.}\frac{x}{2-x\sqrt{3}}+\frac{2}{3}\text{Arc. tang.}\,x+\frac{1}{3}\text{Arc. tang.}\frac{x}{2+x\sqrt{3}}$$

seu

$$y=\frac{1}{3}\text{Arc. tang.}\frac{x}{1-xx}+\frac{2}{3}\text{Arc. tang.}\,x=\frac{1}{3}\text{Arc. tang.}\frac{3x(1-xx)}{1-4xx+x^4}.$$

COROLLARIUM 1

137. Sit

$$z=\int\frac{xxdx}{1+x^6}=\frac{1}{3}x^3-\frac{1}{9}x^9+\frac{1}{15}x^{15}-\frac{1}{21}x^{21}+\text{etc.};$$

at facto $x^3=u$ est

$$z=\frac{1}{3}\int\frac{du}{1+uu}=\frac{1}{3}\text{Arc. tang.}\,u=\frac{1}{3}\text{Arc. tang.}\,x^3.$$

Hinc series huiusmodi mixta formatur

$$x+\frac{n}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7-\frac{n}{9}x^9-\frac{1}{11}x^{11}+\frac{1}{13}x^{13}+\frac{n}{15}x^{15}+\frac{1}{17}x^{17}-\text{etc.},$$

cuius summa est

$$\frac{1}{3}\text{Arc. tang.}\frac{3x(1-xx)}{1-4xx+x^4}+\frac{n}{3}\text{Arc. tang.}\,x^3.$$

COROLLARIUM 2

138. Si hic capiatur $n=-1$, binos angulos in unum colligendo fit

$$\frac{1}{3}\text{Arc. tang.}\frac{3x(1-xx)}{1-4xx+x^4}-\frac{1}{3}\text{Arc. tang.}\,x^3=\frac{1}{3}\text{Arc. tang.}\frac{3x-4x^3+4x^5-x^7}{1-4xx+4x^4-3x^6},$$

quae fractio per $1 - xx + x^4$ dividendo reducitur ad $\frac{3x - x^3}{1 - 3xx}$, quae est tangens tripli anguli x pro tangente habentis, ita ut sit

$$\frac{1}{3}\,\text{Arc. tang.}\,\frac{3x - x^3}{1 - 3xx} = \text{Arc. tang.}\,x,$$

quod idem series inventa manifesto indicat.

EXEMPLUM 6

139. *Hanc formulam* $dy = \frac{(x^{m-1} + x^{n-m-1})\,dx}{1 + x^n}$ *per seriem integrare.*

Ob

$$\frac{1}{1 + x^n} = 1 - x^n + x^{2n} - x^{3n} + x^{4n} - \text{etc.}$$

habebitur

$$y = \frac{x^m}{m} + \frac{x^{n-m}}{n-m} - \frac{x^{m+n}}{n+m} - \frac{x^{2n-m}}{2n-m} + \frac{x^{2n+m}}{2n+m} + \frac{x^{3n-m}}{3n-m} - \text{etc.}$$

Haec ergo series per § 82 aggregatum aliquot arcuum circularium exprimit, quos ibi videre licet.

COROLLARIUM

140. Eodem [modo] proposita formula $dz = \frac{(x^{m-1} - x^{n-m-1})\,dx}{1 - x^n}$ ob

$$\frac{1}{1 - x^n} = 1 + x^n + x^{2n} + x^{3n} + \text{etc.}$$

invenitur

$$z = \frac{x^m}{m} - \frac{x^{n-m}}{n-m} + \frac{x^{n+m}}{n+m} - \frac{x^{2n-m}}{2n-m} + \frac{x^{2n+m}}{2n+m} - \frac{x^{3n-m}}{3n-m} + \text{etc.},$$

cuius valor § 84 est exhibitus.

EXEMPLUM 7

141. *Hanc formulam* $dy = \frac{(1 + 2x)\,dx}{1 + x + xx}$ *per seriem integrare.*

Primo integrale est manifesto $y = l(1 + x + xx)$; ut autem in seriem convertatur, multiplicetur numerator et denominator per $1 - x$, ut fiat

$$dy = \frac{(1 + x - 2xx)\,dx}{1 - x^3}.$$

Cum nunc sit

$$\frac{1}{1-x^3} = 1 + x^3 + x^6 + x^9 + x^{12} + \text{etc.},$$

erit integrando

$$y = x + \frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} - \frac{2x^6}{6} + \frac{x^7}{7} + \frac{x^8}{8} - \frac{2x^9}{9} + \text{etc.}$$

COROLLARIUM 1

142. Eodem modo inveniri potest $y = l(1 + x + xx + x^3)$ per seriem. Cum enim fiat $y + l(1-x) = l(1-x^4)$, erit

$$\begin{aligned} y = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} + \frac{x^8}{8} + \frac{x^9}{9} + \frac{x^{10}}{10} + \text{etc.} \\ - x^4 \qquad\qquad\qquad - \frac{x^8}{2} \qquad\qquad \end{aligned}$$

sive

$$y = x + \frac{x^2}{2} + \frac{x^3}{3} - \frac{3x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} - \frac{3x^8}{8} + \frac{x^9}{9} + \text{etc.}$$

COROLLARIUM 2

143. At fractio $\frac{1+2x}{1+x+xx}$ per seriem recurrentem evoluta dat

$$1 + x - 2xx + x^3 + x^4 - 2x^5 + x^6 + x^7 - 2x^8 + \text{etc.},$$

unde per integrationem eadem series obtinetur quae ante.

EXEMPLUM 8

144. *Hanc formulam* $dy = \frac{dx}{1 - 2x\cos.\zeta + xx}$ *per seriem integrare.*

Per § 64, ubi $A = 1$, $B = 0$, $a = 1$ et $b = 1$, est huius formulae integrale

$$y = \frac{1}{\sin.\zeta}\,\text{Arc. tang.}\,\frac{x\sin.\zeta}{1 - x\cos.\zeta}.$$

At per seriem recurrentem reperimus

$$\begin{aligned} \frac{1}{1 - 2x\cos.\zeta + xx} &= 1 + 2x\cos.\zeta + (4\cos.\zeta^2 - 1)xx + (8\cos.\zeta^3 - 4\cos.\zeta)x^3 \\ &+ (16\cos.\zeta^4 - 12\cos.\zeta^2 + 1)x^4 + (32\cos.\zeta^5 - 32\cos.\zeta^3 + 6\cos.\zeta)x^5 + \text{etc.}, \end{aligned}$$

qua serie per dx multiplicata et integrata obtinetur quaesitum. Potestatibus autem ipsius $\cos.\zeta$ in cosinus angulorum multiplorum conversis reperitur

$$y = x + \frac{1}{2}xx(2\cos.\zeta) + \frac{1}{3}x^3(2\cos.2\zeta + 1) + \frac{1}{4}x^4(2\cos.3\zeta + 2\cos.\zeta)$$
$$+ \frac{1}{5}x^5(2\cos.4\zeta + 2\cos.2\zeta + 1) + \frac{1}{6}x^6(2\cos.5\zeta + 2\cos.3\zeta + 2\cos.\zeta) + \text{etc.}$$

COROLLARIUM 1

145. Si ponatur
$$dz = \frac{(1 - x\cos.\zeta)dx}{1 - 2x\cos.\zeta + xx},$$
erit per § 63 $A = 1$, $B = -\cos.\zeta$, $a = 1$ et $b = 1$ ideoque
$$z = -\cos.\zeta\, l\sqrt{(1 - 2x\cos.\zeta + xx)} + \sin.\zeta \text{ Arc. tang.} \frac{x\sin.\zeta}{1 - x\cos.\zeta};$$
at per seriem ob
$$\frac{1 - x\cos.\zeta}{1 - 2x\cos.\zeta + xx} = 1 + x\cos.\zeta + x^2\cos.2\zeta + x^3\cos.3\zeta + x^4\cos.4\zeta + \text{etc.}$$
fit
$$z = x + \frac{1}{2}xx\cos.\zeta + \frac{1}{3}x^3\cos.2\zeta + \frac{1}{4}x^4\cos.3\zeta + \frac{1}{5}x^5\cos.4\zeta + \text{etc.}$$

COROLLARIUM 2

146. At quia
$$dz = \frac{dx(-x\cos.\zeta + \cos.\zeta^2 + \sin.\zeta^2)}{1 - 2x\cos.\zeta + xx},$$
erit
$$z = -\cos.\zeta\, l\sqrt{(1 - 2x\cos.\zeta + xx)} + \sin.\zeta^2 \int \frac{dx}{1 - 2x\cos.\zeta + xx}.$$
Hinc ergo pro
$$y = \int \frac{dx}{1 - 2x\cos.\zeta + xx}$$
alia reperitur series infinita cum logarithmo connexa, scilicet
$$y = \frac{\cos.\zeta}{\sin.\zeta^2}\, l\sqrt{(1 - 2x\cos.\zeta + xx)}$$
$$+ \frac{1}{\sin.\zeta^2}\left(x + \frac{1}{2}xx\cos.\zeta + \frac{1}{3}x^3\cos.2\zeta + \frac{1}{4}x^4\cos.3\zeta + \text{etc.}\right).$$

PROBLEMA 12[a][1])

147. *Formulam differentialem irrationalem*

$$dy = x^{m-1}dx(a + bx^n)^{\frac{\mu}{\nu}}$$

per seriem infinitam integrare.

SOLUTIO

Sit $a^{\frac{\mu}{\nu}} = c$; erit

$$dy = cx^{m-1}dx\left(1 + \frac{b}{a}x^n\right)^{\frac{\mu}{\nu}},$$

ubi quidem assumimus c non esse quantitatem imaginariam. Cum igitur sit

$$\left(1 + \frac{b}{a}x^n\right)^{\frac{\mu}{\nu}} = 1 + \frac{\mu b}{1\nu \cdot a}x^n + \frac{\mu(\mu-\nu)bb}{1\nu \cdot 2\nu \cdot aa}x^{2n} + \frac{\mu(\mu-\nu)(\mu-2\nu)b^3}{1\nu \cdot 2\nu \cdot 3\nu \cdot a^3}x^{3n} + \text{etc.},$$

erit integrando

$$y = c\left(\frac{x^m}{m} + \frac{\mu b}{1\nu \cdot a} \cdot \frac{x^{m+n}}{m+n} + \frac{\mu(\mu-\nu)bb}{1\nu \cdot 2\nu \cdot aa} \cdot \frac{x^{m+2n}}{m+2n} + \frac{\mu(\mu-\nu)(\mu-2\nu)b^3}{1\nu \cdot 2\nu \cdot 3\nu \cdot a^3} \cdot \frac{x^{m+3n}}{m+3n} + \text{etc.}\right),$$

quae series in infinitum excurrit, nisi $\frac{\mu}{\nu}$ sit numerus integer positivus.

Sin autem casu, quo ν numerus par, a fuerit quantitas negativa, expressio nostra ita est repraesentanda

$$dy = x^{m-1}dx(bx^n - a)^{\frac{\mu}{\nu}} = b^{\frac{\mu}{\nu}}x^{m+\frac{\mu n}{\nu}-1}dx\left(1 - \frac{a}{b}x^{-n}\right)^{\frac{\mu}{\nu}}.$$

Cum igitur sit

$$\left(1 - \frac{a}{b}x^{-n}\right)^{\frac{\mu}{\nu}} = 1 - \frac{\mu a}{1\nu \cdot b}x^{-n} + \frac{\mu(\mu-\nu)a^2}{1\nu \cdot 2\nu \cdot b^2}x^{-2n} - \frac{\mu(\mu-\nu)(\mu-2\nu)a^3}{1\nu \cdot 2\nu \cdot 3\nu \cdot b^3}x^{-3n} + \text{etc.},$$

erit integrando

$$y = b^{\frac{\mu}{\nu}}\left(\frac{\nu x^{m+\frac{\mu n}{\nu}}}{m\nu + \mu n} - \frac{\mu a}{1\nu \cdot b} \cdot \frac{\nu x^{m+\frac{(\mu-\nu)n}{\nu}}}{m\nu + (\mu-\nu)n} + \frac{\mu(\mu-\nu)a^2}{1\nu \cdot 2\nu \cdot b^2} \cdot \frac{\nu x^{m+\frac{(\mu-2\nu)n}{\nu}}}{m\nu + (\mu-2\nu)n} - \text{etc.}\right).$$

Si a et b sint numeri positivi, utraque evolutione uti licet.

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1) In editione principe falso numerus 12 iteratur. L. S.

EXEMPLUM 1

148. *Formulam* $dy = \frac{dx}{\sqrt{(1-xx)}}$ *per seriem integrare.*

Primo ex superioribus patet esse $y = \text{Arc.}\sin. x$, qui ergo angulus etiam per seriem infinitam exprimetur. Cum enim sit

$$\frac{1}{\sqrt{(1-xx)}} = 1 + \frac{1}{2}x^2 + \frac{1\cdot 3}{2\cdot 4}x^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}x^8 + \text{etc.},$$

erit

$$y = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^7}{7} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{x^9}{9} + \text{etc.}$$

utroque valore ita definito, ut evanescat posito $x = 0$.

COROLLARIUM 1

149. Si ergo sit $x = 1$, ob $\text{Arc.}\sin. 1 = \frac{\pi}{2}$ erit

$$\frac{\pi}{2} = 1 + \frac{1}{2\cdot 3} + \frac{1\cdot 3}{2\cdot 4\cdot 5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 9} + \text{etc.}$$

At si ponatur $x = \frac{1}{2}$, ob $\text{Arc.}\sin. \frac{1}{2} = 30^0 = \frac{\pi}{6}$ erit

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2\cdot 2^3\cdot 3} + \frac{1\cdot 3}{2\cdot 4\cdot 2^5\cdot 5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 2^7\cdot 7} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 2^9\cdot 9} + \text{etc.},$$

cuius seriei decem termini additi dant 0,52359877, cuius sextuplum 3,14159262 tantum in octava figura a veritate discrepat.

COROLLARIUM 2

150. Proposita hac formula $dy = \frac{dx}{\sqrt{(x-xx)}}$ posito $x = uu$ fit

$$dy = \frac{2udu}{\sqrt{(uu-u^4)}} = \frac{2du}{\sqrt{(1-uu)}},$$

ergo

$$y = 2\,\text{Arc.}\sin. u = 2\,\text{Arc.}\sin. \sqrt{x};$$

tum vero per seriem erit

$$y = 2\left(u + \frac{1}{2}\cdot\frac{u^3}{3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{u^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{u^7}{7} + \text{etc.}\right)$$

seu

$$y = 2\left(1 + \frac{1}{2}\cdot\frac{x}{3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{xx}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^3}{7} + \text{etc.}\right)\sqrt{x}.$$

EXEMPLUM 2

151. *Formulam $dy = dx\sqrt{(2ax - xx)}$ per seriem integrare.*

Posito $x = uu$ fit $dy = 2uudu\sqrt{(2a - uu)}$; at per reductionem I (§ 118) est $n = 2$, $m = 1$, $a = 2a$, $b = -1$, $\mu = 1$, $\nu = 2$, unde

$$\int uudu\sqrt{(2a - uu)} = -\frac{1}{4}u(2a - uu)^{\frac{3}{2}} + \frac{1}{2}a\int du\sqrt{(2a - uu)},$$

et per III sumendo $m = 1$, $a = 2a$, $b = -1$, $n = 2$, $\mu = -1$, $\nu = 2$ fit

$$\int du\sqrt{(2a - uu)} = \frac{1}{2}u\sqrt{(2a - uu)} + a\int\frac{du}{\sqrt{(2a - uu)}};$$

at est

$$\int\frac{du}{\sqrt{(2a - uu)}} = \text{Arc. sin.}\frac{u}{\sqrt{2a}} = \text{Arc. sin.}\frac{\sqrt{x}}{\sqrt{2a}}$$

ideoque

$$\int uudu\sqrt{(2a - uu)} = -\frac{1}{4}u(2a - uu)^{\frac{3}{2}} + \frac{1}{4}au\sqrt{(2a - uu)} + \frac{1}{2}aa\,\text{Arc. sin.}\frac{\sqrt{x}}{\sqrt{2a}}$$

$$= \frac{1}{4}u(uu - a)\sqrt{(2a - uu)} + \frac{1}{2}aa\,\text{Arc. sin.}\frac{\sqrt{x}}{\sqrt{2a}}.$$

Ergo

$$y = \frac{1}{2}(x - a)\sqrt{(2ax - xx)} + aa\,\text{Arc. sin.}\frac{\sqrt{x}}{\sqrt{2a}}.$$

Pro serie autem invenienda est

$$dy = dx\sqrt{2ax}\left(1 - \frac{x}{2a}\right)^{\frac{1}{2}}$$

$$= x^{\frac{1}{2}}dx\left(1 - \frac{1}{2}\cdot\frac{x}{2a} - \frac{1\cdot 1}{2\cdot 4}\cdot\frac{xx}{4aa} - \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\cdot\frac{x^3}{8a^3} - \text{etc.}\right)\sqrt{2a}$$

hincque integrando

$$y = \left(\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{2}\cdot\frac{2x^{\frac{5}{2}}}{5\cdot 2a} - \frac{1\cdot 1}{2\cdot 4}\cdot\frac{2x^{\frac{7}{2}}}{7\cdot 4aa} - \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\cdot\frac{2x^{\frac{9}{2}}}{9\cdot 8a^3} - \text{etc.}\right)\sqrt{2a}$$

seu

$$y = \left(\frac{x}{3} - \frac{1}{2}\cdot\frac{x^2}{5\cdot 2a} - \frac{1\cdot 1}{2\cdot 4}\cdot\frac{x^3}{7\cdot 4aa} - \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\cdot\frac{x^4}{9\cdot 8a^3} - \text{etc.}\right)2\sqrt{2ax}.$$

COROLLARIUM 1

152. Integrale facilius inveniri potest ponendo $x = a - v$, unde fit

$$dy = -dv\sqrt{(aa - vv)}$$

et per reductionem III [§ 118]

$$\int dv\sqrt{(aa-vv)} = \frac{1}{2}v\sqrt{(aa-vv)} + \frac{1}{2}aa\int\frac{dv}{\sqrt{(aa-vv)}},$$

hinc

$$y = C - \frac{1}{2}v\sqrt{(aa-vv)} - \frac{1}{2}aa\,\text{Arc.sin.}\frac{v}{a}$$

seu

$$y = C - \frac{1}{2}(a-x)\sqrt{(2ax-xx)} - \frac{1}{2}aa\,\text{Arc.sin.}\frac{a-x}{a};$$

ut igitur fiat $y = 0$ posito $x = 0$, capi debet $C = \frac{1}{2}aa\,\text{Arc.sin.}1$, ita ut sit

$$y = -\frac{1}{2}(a-x)\sqrt{(2ax-xx)} + \frac{1}{2}aa\,\text{Arc.cos.}\frac{a-x}{a}.$$

Est vero

$$\text{Arc.sin.}\frac{\sqrt{x}}{\sqrt{2a}} = \frac{1}{2}\text{Arc.cos.}\frac{a-x}{a}.$$

COROLLARIUM 2

153. Si ponamus $x = \frac{a}{2}$, fit $y = \frac{-aa\sqrt{3}}{8} + \frac{\pi aa}{6}$; series autem dat

$$y = 2aa\left(\frac{1}{2\cdot 3} - \frac{1}{2\cdot 5\cdot 2^3} - \frac{1\cdot 1}{2\cdot 4\cdot 7\cdot 2^5} - \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6\cdot 9\cdot 2^7} - \text{etc.}\right),$$

unde colligitur

$$\pi = \frac{3\sqrt{3}}{4} + 6\left(\frac{1}{3} - \frac{1}{2\cdot 5\cdot 2^2} - \frac{1\cdot 1}{2\cdot 4\cdot 7\cdot 2^4} - \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6\cdot 9\cdot 2^6} - \text{etc.}\right);$$

at per superiorem [§ 149] est

$$\pi = 3\left(1 + \frac{1}{2\cdot 3\cdot 2^2} + \frac{1\cdot 3}{2\cdot 4\cdot 5\cdot 2^4} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7\cdot 2^6} + \text{etc.}\right),$$

ex quarum combinatione plures aliae formari possunt.

EXEMPLUM 3

154. *Formulam* $dy = \frac{dx}{\sqrt{(1+xx)}}$ *per seriem integrare.*

Integrale est $y = l\left(x + \sqrt{(1+xx)}\right)$ ita sumtum, ut evanescat posito $x = 0$. At ob

$$\frac{1}{\sqrt{(1+xx)}} = 1 - \frac{1}{2}x^2 + \frac{1\cdot 3}{2\cdot 4}x^4 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 + \text{etc.}$$

erit idem integrale per seriem expressum

$$y = x - \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^7}{7} + \text{etc.}$$

EXEMPLUM 4

155. *Formulam* $dy = \frac{dx}{\sqrt{(xx-1)}}$ *per seriem integrare.*

Integratio dat $y = l\left(x + \sqrt{(xx-1)}\right)$, quod evanescit posito $x = 1$. Iam ob

$$\frac{1}{\sqrt{(xx-1)}} = \frac{1}{x} + \frac{1}{2x^3} + \frac{1\cdot 3}{2\cdot 4x^5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6x^7} + \text{etc.}$$

erit idem integrale

$$y = C + lx - \frac{1}{2\cdot 2x^2} - \frac{1\cdot 3}{2\cdot 4\cdot 4x^4} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 6x^6} - \text{etc.};$$

quod ut evanescat posito $x = 1$, constans ita definitur, ut fiat

$$y = lx + \frac{1}{2\cdot 2}\left(1 - \frac{1}{xx}\right) + \frac{1\cdot 3}{2\cdot 4\cdot 4}\left(1 - \frac{1}{x^4}\right) + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 6}\left(1 - \frac{1}{x^6}\right) + \text{etc.}$$

COROLLARIUM

156. Posito $x = 1 + u$ fit

$$dy = \frac{du}{\sqrt{(2u+uu)}} = \frac{du}{\sqrt{2u}}\left(1+\frac{u}{2}\right)^{-\frac{1}{2}}$$
$$= \frac{du}{\sqrt{2u}}\left(1 - \frac{1}{2}\cdot\frac{u}{2} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{uu}{4} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{u^3}{8} - \text{etc.}\right),$$

unde integrando habebitur

$$y = \frac{1}{\sqrt{2}}\left(2\sqrt{u} - \frac{1}{2}\cdot\frac{2u^{\frac{3}{2}}}{3\cdot 2} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{2u^{\frac{5}{2}}}{5\cdot 4} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{2u^{\frac{7}{2}}}{7\cdot 8} + \text{etc.}\right)$$

seu

$$y = \left(1 - \frac{1u}{2\cdot 3\cdot 2} + \frac{1\cdot 3uu}{2\cdot 4\cdot 5\cdot 4} - \frac{1\cdot 3\cdot 5u^3}{2\cdot 4\cdot 6\cdot 7\cdot 8} + \text{etc.}\right)\sqrt{2u}.$$

EXEMPLUM 5

157. *Formulam* $dy = \frac{dx}{(1-x)^n}$ *per seriem integrare.*

Per integrationem fit

$$y = \frac{1}{(n-1)(1-x)^{n-1}} - \frac{1}{n-1}$$

facto $y = 0$, si $x = 0$, seu

$$y = \frac{(1-x)^{-n+1} - 1}{n-1}.$$

Iam vero per seriem est

$$dy = dx\left(1 + nx + \frac{n(n+1)}{1\cdot 2}x^2 + \frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}x^3 + \text{etc.}\right),$$

unde idem integrale ita exprimetur

$$y = x + \frac{nx^2}{2} + \frac{n(n+1)x^3}{1\cdot 2\cdot 3} + \frac{n(n+1)(n+2)x^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.}$$

Hinc autem quoque manifesto fit

$$(n-1)y + 1 = \frac{1}{(1-x)^{n-1}}.$$

SCHOLION

158. Haec autem cum sint nimis obvia, quam ut iis fusius inhaerere sit opus, aliam methodum series eliciendi exponam magis absconditam, quae saepe in Analysi eximium usum afferre potest.

PROBLEMA 13

159. *Proposita formula differentiali*

$$dy = x^{m-1}dx(a+bx^n)^{\frac{\mu}{\nu}-1}$$

eius integrale altera methodo in seriem convertere.

SOLUTIO

Ponatur $y=(a+bx^n)^{\frac{\mu}{\nu}}z$; erit

$$dy=(a+bx^n)^{\frac{\mu}{\nu}-1}\left(dz(a+bx^n)+\frac{n\mu}{\nu}bx^{n-1}zdx\right),$$

unde fit

$$x^{m-1}dx = dz(a+bx^n)+\frac{n\mu}{\nu}bx^{n-1}zdx$$

seu

$$\nu x^{m-1}dx=\nu dz(a+bx^n)+n\mu bx^{n-1}zdx.$$

Iam antequam seriem, qua valor ipsius z definiatur, investigemus, notandum est casu, quo x evanescit, fieri

$$dy=a^{\frac{\mu}{\nu}-1}x^{m-1}dx=a^{\frac{\mu}{\nu}}dz,$$

ut sit $dz=\frac{1}{a}x^{m-1}dx$. Statuamus ergo

$$z=Ax^m+Bx^{m+n}+Cx^{m+2n}+Dx^{m+3n}+\text{etc.}$$

eritque

$$\frac{dz}{dx}=mAx^{m-1}+(m+n)Bx^{m+n-1}+(m+2n)Cx^{m+2n-1}+\text{etc.}$$

Substituantur hae series loco z et $\frac{dz}{dx}$ in aequatione

$$\frac{\nu dz}{dx}(a+bx^n)+n\mu bx^{n-1}z-\nu x^{m-1}=0$$

singulisque terminis secundum potestates ipsius x dispositis orietur ista aequatio

$$\left.\begin{array}{llll} m\nu aAx^{m-1} & +(m+n)\nu aBx^{m+n-1} & +(m+2n)\nu aCx^{m+2n-1} & +\text{etc.} \\ -\nu & +\quad m\nu bA & +\quad (m+n)\nu bB & \\ & +\quad n\mu bA & +\quad n\mu bB & \end{array}\right\}=0,$$

unde singulis terminis nihilo aequalibus positis coefficientes ficti per sequentes formulas definientur

$$m\nu aA-\nu=0, \qquad \text{hinc} \quad A=\frac{1}{ma},$$

$$(m+n)\nu aB+(m\nu+n\mu)bA=0, \qquad B=-\frac{(m\nu+n\mu)b}{(m+n)\nu a}A,$$

$$(m+2n)\nu aC+((m+n)\nu+n\mu)bB=0, \qquad C=-\frac{((m+n)\nu+n\mu)b}{(m+2n)\nu a}B,$$

$$(m+3n)\nu aD+((m+2n)\nu+n\mu)bC=0, \qquad D=-\frac{((m+2n)\nu+n\mu)b}{(m+3n)\nu a}C$$

sicque quilibet coefficiens facile ex praecedente reperitur. Tum vero erit

$$y=(a+bx^n)^{\frac{\mu}{\nu}}(Ax^m+Bx^{m+n}+Cx^{m+2n}+Dx^{m+3n}+\text{etc.}).$$

SOLUTIO 2

Quemadmodum hic seriem secundum potestates ipsius x ascendentem assumsimus, ita etiam descendentem constituere licet

$$z=Ax^{m-n}+Bx^{m-2n}+Cx^{m-3n}+Dx^{m-4n}+\text{etc.},$$

ut sit

$$\frac{dz}{dx}=(m-n)Ax^{m-n-1}+(m-2n)Bx^{m-2n-1}+(m-3n)Cx^{m-3n-1}+\text{etc.},$$

quibus seriebus substitutis prodit

$$\left.\begin{array}{lllll} (m-n)\nu bAx^{m-1} & +(m-n)\nu aAx^{m-n-1} & +(m-2n)\nu aBx^{m-2n-1} & +(m-3n)\nu aCx^{m-3n-1} & +\text{etc.} \\ +n\mu bA & +(m-2n)\nu bB & +(m-3n)\nu bC & +(m-4n)\nu bD & \\ -\nu & +\quad n\mu bB & +\quad n\mu bC & +\quad n\mu bD & \end{array}\right\}=0.$$

Hinc ergo sequenti modo litterae A, B, C etc. determinantur

$$(m-n)\nu bA + n\mu bA - \nu = 0, \qquad \text{ergo} \quad A = \frac{\nu}{(m-n)\nu + n\mu} \cdot \frac{1}{b},$$

$$(m-n)\nu aA + (m-2n)\nu bB + n\mu bB = 0, \qquad B = \frac{-(m-n)\nu}{(m-2n)\nu + n\mu} \cdot \frac{a}{b} A,$$

$$(m-2n)\nu aB + (m-3n)\nu bC + n\mu bC = 0, \qquad C = \frac{-(m-2n)\nu}{(m-3n)\nu + n\mu} \cdot \frac{a}{b} B,$$

$$(m-3n)\nu aC + (m-4n)\nu bD + n\mu bD = 0, \qquad D = \frac{-(m-3n)\nu}{(m-4n)\nu + n\mu} \cdot \frac{a}{b} C,$$

ubi iterum lex progressionis harum litterarum est manifesta.

COROLLARIUM 1

160. Prior series ideo est memorabilis, quod casibus, quibus

$$(m+in)\nu + n\mu = 0 \quad \text{seu} \quad -\frac{m}{n} - \frac{\mu}{\nu} = i,$$

abrumpitur atque ipsum integrale algebraicum exhibet. Posterior vero abrumpitur, quoties $m - in = 0$ seu $\frac{m}{n} = i$ denotante i numerum integrum positivum.

COROLLARIUM 2

161. Utraque vero series etiam incommodo quodam laborat, quod non semper in usum vocari potest. Quando enim vel $m = 0$ vel $m + in = 0$, priori uti non licet, quando vero $(m - in)\nu + n\mu = 0$ seu $\frac{m}{n} + \frac{\mu}{\nu} = i$, usus posterioris tollitur, quia termini fierent infiniti.

COROLLARIUM 3

162. Hoc vero commode usu venit, ut, quoties altera applicari nequit, altera certo in usum vocari possit, iis tantum casibus exceptis, quibus et $-\frac{m}{n}$ et $\frac{\mu}{\nu} + \frac{m}{n}$ sunt numeri integri positivi. Quia autem tum est $\nu = 1$, hi casus sunt rationales integri nihilque difficultatis habent.

COROLLARIUM 4

163. Possunt etiam ambae series simul pro z coniungi hoc modo. Sit prior series $= P$, posterior vero $= Q$, ut capi possit tam $z = P$ quam $z = Q$. Binis autem coniungendis erit $z = \alpha P + \beta Q$, dummodo sit $\alpha + \beta = 1$.

SCHOLION

164. Inde autem, quod duas series pro z exhibemus, minime sequitur has duas series inter se esse aequales; neque enim necesse est, ut valores ipsius y inde orti fiant aequales, dummodo quantitate constante a se invicem differant. Ita si prior series inventa per P, posterior per Q indicetur, quia ex illa fit $y = (a + bx^n)^{\frac{\mu}{\nu}} P$, ex hac vero $y = (a + bx^n)^{\frac{\mu}{\nu}} Q$, certo erit $(a + bx^n)^{\frac{\mu}{\nu}}(P - Q)$ quantitas constans ideoque $P - Q = C(a + bx^n)^{-\frac{\mu}{\nu}}$. Utraque scilicet series tantum integrale particulare praebet, quoniam nullam constantem involvit, quae non iam in formula differentiali contineatur. Interim tamen eadem methodo etiam valor completus pro z erui potest; praeter seriem enim assumtam P vel Q statui potest

$$z = P + \alpha + \beta x^n + \gamma x^{2n} + \delta x^{3n} + \varepsilon x^{4n} + \text{etc.}$$

ac substitutione facta series P ut ante definitur; pro altera vero nova serie efficiendum est, ut sit

$$\left.\begin{array}{lllll} n\nu\alpha\beta x^{n-1} & + 2n\nu\alpha\gamma x^{2n-1} & + 3n\nu\alpha\delta x^{3n-1} & + 4n\nu\alpha\varepsilon x^{4n-1} & + \text{etc.} \\ + n\mu b\alpha & + \ n\nu b\beta & + 2n\nu b\gamma & + 3n\nu b\delta & \\ & + \ n\mu b\beta & + \ n\mu b\gamma & + \ n\mu b\delta & \end{array}\right\} = 0,$$

unde ducuntur hae determinationes

$$\beta = \frac{-\mu b}{\nu a}\cdot\alpha, \quad \gamma = \frac{-(\mu+\nu)b}{2\nu a}\cdot\beta, \quad \delta = \frac{-(\mu+2\nu)b}{3\nu a}\cdot\gamma, \quad \varepsilon = \frac{-(\mu+3\nu)b}{4\nu a}\cdot\delta \ \text{ etc.},$$

ita ut prodeat

$$z = P + \alpha\left(1 - \frac{\mu}{\nu}\cdot\frac{b}{a}x^n + \frac{\mu(\mu+\nu)}{\nu\cdot 2\nu}\cdot\frac{b^2}{a^2}x^{2n} - \frac{\mu(\mu+\nu)(\mu+2\nu)}{\nu\cdot 2\nu\cdot 3\nu}\cdot\frac{b^3}{a^3}x^{3n} + \text{etc.}\right)$$

seu

$$z = P + \alpha\left(1 + \frac{b}{a}x^n\right)^{-\frac{\mu}{\nu}}$$

hincque

$$y = P(a + bx^n)^{\frac{\mu}{\nu}} + \alpha a^{\frac{\mu}{\nu}},$$

quod est integrale completum, quia constans α mansit arbitraria.

EXEMPLUM 1

165. *Formulam* $dy = \frac{dx}{\sqrt{(1-xx)}}$ *hoc modo per seriem integrare.*

Comparatione cum forma generali instituta fit $a = 1$, $b = -1$, $m = 1$, $n = 2$, $\mu = 1$, $\nu = 2$, unde posito $y = z\sqrt{(1-xx)}$ prima solutio

$$z = Ax + Bx^3 + Cx^5 + Dx^7 + \text{etc.}$$

praebet

$$A = 1, \quad B = \frac{2}{3}A, \quad C = \frac{4}{5}B, \quad D = \frac{6}{7}C, \quad E = \frac{8}{9}D \quad \text{etc.},$$

unde colligimus

$$y = \left(x + \frac{2}{3}x^3 + \frac{2\cdot4}{3\cdot5}x^5 + \frac{2\cdot4\cdot6}{3\cdot5\cdot7}x^7 + \text{etc.}\right)\sqrt{(1-xx)},$$

quod integrale evanescit posito $x = 0$; est ergo $y = \text{Arc.sin.}\,x$. Altera methodus hic frustra tentatur ob $\frac{m}{n} + \frac{\mu}{\nu} = 1$.

COROLLARIUM 1

166. Posito $x = 1$ videtur hinc fieri $y = 0$ ob $\sqrt{(1-xx)} = 0$; at perpendendum est fieri hoc casu seriei infinitae summam infinitam, ita ut nihil obstet, quominus sit $y = \frac{\pi}{2}$. Si ponamus $x = \frac{1}{2}$, fit $y = 30^0 = \frac{\pi}{6}$ ideoque

$$\frac{\pi}{6} = \left(1 + \frac{2}{3\cdot4} + \frac{2\cdot4}{3\cdot5\cdot4^2} + \frac{2\cdot4\cdot6}{3\cdot5\cdot7\cdot4^3} + \text{etc.}\right)\frac{\sqrt{3}}{4}.$$

COROLLARIUM 2

167. Simili modo proposita formula $dy = \frac{dx}{\sqrt{(1+xx)}}$ reperitur

$$y = \left(x - \frac{2}{3}x^3 + \frac{2\cdot 4}{3\cdot 5}x^5 - \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}x^7 + \text{etc.}\right)\sqrt{(1+xx)}$$

estque

$$y = l\left(x + \sqrt{(1+xx)}\right).$$

EXEMPLUM 2

168. *Formulam* $dy = \frac{dx}{x\sqrt{(1-xx)}}$ *hoc modo per seriem integrare.*

Est ergo $m = 0$, $n = 2$, $\mu = 1$, $\nu = 2$, $a = 1$ et $b = -1$; utendum igitur est altera serie sumendo

$$z = \frac{y}{\sqrt{(1-xx)}} = Ax^{-2} + Bx^{-4} + Cx^{-6} + Dx^{-8} + \text{etc.}$$

fitque

$$A = 1, \quad B = \frac{2}{3}A, \quad C = \frac{4}{5}B, \quad D = \frac{6}{7}C \quad \text{etc.}$$

Hinc ergo colligimus

$$y = \left(\frac{1}{xx} + \frac{2}{3x^4} + \frac{2\cdot 4}{3\cdot 5x^6} + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7x^8} + \text{etc.}\right)\sqrt{(1-xx)}.$$

At integratio praebet

$$y = l\frac{1-\sqrt{(1-xx)}}{x},$$

qui valores conveniunt, quia uterque evanescit posito $x = 1$.[1])

COROLLARIUM 1

169. Cum autem haec series non convergat, nisi capiatur $x > 1$, hoc autem casu formula $\sqrt{(1-xx)}$ fiat imaginaria, haec series nullius est usus.

1) Cum posito $x = 1$ seriei summa infinita evadat, pro hoc ipsius x valore productum seriei in $\sqrt{(1-xx)}$ evanescere non sequitur; cf. § 166. L. S.

COROLLARIUM 2

170. Si proponatur $dy = \frac{dx}{x\sqrt{(xx-1)}}$, eadem pro y series emergit per $\sqrt{-1}$ multiplicata eritque

$$y = -\left(\frac{1}{xx} + \frac{2}{3x^4} + \frac{2\cdot 4}{3\cdot 5x^6} + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7x^8} + \text{etc.}\right)\sqrt{(xx-1)}.$$

Posito autem $x = \frac{1}{u}$ erit $dy = \frac{-du}{\sqrt{(1-uu)}}$ et $y = C - \text{Arc.sin.}\,u$ seu

$$y = C - \text{Arc.sin.}\,\frac{1}{x},$$

ubi sumi oportet $C = 0$, quia series illa evanescit posito $x = \infty$, ita ut sit $y = -\text{Arc.sin.}\,\frac{1}{x}$, quae cum superiori [§ 165] convenit statuendo $\frac{1}{x} = u$.

EXEMPLUM 3

171. *Formulam* $dy = \frac{dx}{\sqrt{(a+bx^4)}}$ *hoc modo per seriem integrare.*

Est hic $m = 1$, $n = 4$, $\mu = 1$, $\nu = 2$ ideoque posito $y = z\sqrt{(a+bx^4)}$ prior resolutio dat

$$z = Ax + Bx^5 + Cx^9 + Dx^{13} + \text{etc.}$$

existente

$$A = \frac{1}{a}, \quad B = \frac{-3b}{5a}A, \quad C = \frac{-7b}{9a}B, \quad D = \frac{-11b}{13a}C \text{ etc.},$$

ita ut sit

$$y = \left(\frac{x}{a} - \frac{3bx^5}{5aa} + \frac{3\cdot 7b^2x^9}{5\cdot 9a^3} - \frac{3\cdot 7\cdot 11b^3x^{13}}{5\cdot 9\cdot 13a^4} + \text{etc.}\right)\sqrt{(a+bx^4)}.$$

Hic autem quoque altera resolutio locum habet ponendo

$$z = Ax^{-3} + Bx^{-7} + Cx^{-11} + Dx^{-15} + \text{etc.}$$

existente

$$A = \frac{-1}{b}, \quad B = \frac{-3a}{5b}A, \quad C = \frac{-7a}{9b}B, \quad D = \frac{-11a}{13b}C \text{ etc.},$$

unde colligitur

$$y = -\left(\frac{1}{bx^3} - \frac{3a}{5b^2x^7} + \frac{3\cdot 7aa}{5\cdot 9b^3x^{11}} - \frac{3\cdot 7\cdot 11a^3}{5\cdot 9\cdot 13b^4x^{15}} + \text{etc.}\right)\sqrt{(a+bx^4)},$$

quarum serierum illa evanescit posito $x = 0$, haec vero posito $x = \infty$.

COROLLARIUM 1

172. Differentia ergo harum duarum serierum est constans, scilicet

$$\left\{\begin{aligned}&+\frac{x}{a}-\frac{3bx^5}{5aa}+\frac{3\cdot 7b^2x^9}{5\cdot 9a^3}-\frac{3\cdot 7\cdot 11b^3x^{13}}{5\cdot 9\cdot 13a^4}+\text{etc.}\\&+\frac{1}{bx^3}-\frac{3a}{5bbx^7}+\frac{3\cdot 7a^2}{5\cdot 9b^3x^{11}}-\frac{3\cdot 7\cdot 11a^3}{5\cdot 9\cdot 13b^4x^{15}}+\text{etc.}\end{aligned}\right\}\sqrt{(a+bx^4)}=\text{Const.}\ ^{1)}$$

COROLLARIUM 2

173. Has ergo binas series colligendo habebimus

$$\frac{a+bx^4}{abx^3}-\frac{3}{5}\cdot\frac{a^3+b^3x^{12}}{a^2b^2x^7}+\frac{3\cdot 7}{5\cdot 9}\cdot\frac{a^5+b^5x^{20}}{a^3b^3x^{11}}-\text{etc.}=\frac{C}{\sqrt{(a+bx^4)}},$$

ubi, quicunque valor ipsi x tribuatur, pro C semper eadem quantitas obtinetur.

COROLLARIUM 3

174. Ita si $a=1$ et $b=1$, erit haec series in $\sqrt{(1+x^4)}$ ducta semper constans, scilicet

$$\left(\frac{1+x^4}{x^3}-\frac{3}{5}\cdot\frac{1+x^{12}}{x^7}+\frac{3\cdot 7}{5\cdot 9}\cdot\frac{1+x^{20}}{x^{11}}-\text{etc.}\right)\sqrt{(1+x^4)}=C.$$

Cum igitur posito $x=1$ fiat

$$C=\left(1-\frac{3}{5}+\frac{3\cdot 7}{5\cdot 9}-\frac{3\cdot 7\cdot 11}{5\cdot 9\cdot 13}+\text{etc.}\right)2\sqrt{2},$$

huicque valori etiam illa series, quicunque valor ipsi x tribuatur, est aequalis.

COROLLARIUM 4

175. Haec postrema series signis alternantibus procedens per differentias facile in aliam iisdem signis praeditam transformatur, unde eadem constans

1) Series hac et sequente paragrapho consideratae non simul convergunt, nisi valor absolutus ipsius x aequalis sit valori absoluto quantitatis $\sqrt[4]{\frac{a}{b}}$ non evanescente $a+bx^4$. L. S.

concluditur

$$C = \left(1 + \frac{1}{5} + \frac{1\cdot 3}{5\cdot 9} + \frac{1\cdot 3\cdot 5}{5\cdot 9\cdot 13} + \frac{1\cdot 3\cdot 5\cdot 7}{5\cdot 9\cdot 13\cdot 17} + \text{etc.}\right)\sqrt{2},$$

quae series satis cito convergit, eritque proxime $C = \frac{13}{7}$.

SCHOLION

176. Ista methodus in hoc consistit, ut series quaedam indefinita fingatur eiusque determinatio ex natura rei derivetur. Eius usus autem potissimum cernitur in aequationibus differentialibus resolvendis; verum etiam in praesenti instituto saepe utiliter adhibetur. Eiusdem quoque methodi ope quantitates transcendentes reciprocae, veluti exponentiales et sinus cosinusve angulorum, per series exprimuntur; quae etsi iam aliunde sint cognitae, tamen earum investigationem per integrationem exposuisse iuvabit, cum simili modo alia praeclara erui queant.

PROBLEMA 14

177. *Quantitatem exponentialem $y = a^x$ in seriem convertere.*

SOLUTIO

Sumtis logarithmis habemus $ly = xla$ et differentiando

$$\frac{dy}{y} = dxla \quad \text{seu} \quad \frac{dy}{dx} = yla,$$

unde valorem ipsius y per seriem quaeri oportet. Cum autem integrale completum latius pateat, notetur nostro casu posito $x = 0$ fieri debere $y = 1$; quare fingatur haec pro y series

$$y = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + \text{etc.},$$

unde fit

$$\frac{dy}{dx} = A + 2Bx + 3Cx^2 + 4Dx^3 + \text{etc.},$$

quibus substitutis in aequatione $\frac{dy}{dx} - yla = 0$ erit

$$\left.\begin{array}{llllll} A + & 2Bx + & 3Cx^2 + & 4Dx^3 + & 5Ex^4 + & \text{etc.} \\ -la - & Ala & -Bla & -Cla & -Dla & \end{array}\right\} = 0$$

hincque coefficientes ita determinantur

$$A = la, \quad B = \frac{1}{2}Ala, \quad C = \frac{1}{3}Bla, \quad D = \frac{1}{4}Cla \quad \text{etc.}$$

sicque consequimur

$$y = a^x = 1 + \frac{xla}{1} + \frac{x^2(la)^2}{1\cdot 2} + \frac{x^3(la)^3}{1\cdot 2\cdot 3} + \frac{x^4(la)^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.},$$

quae est ipsa series notissima in *Introductione*[1]) data.

SCHOLION

178. Pro sinibus et cosinibus angulorum ad differentialia secundi gradus est descendendum, ex quibus deinceps series integrale referens elici debet. Cum autem gemina integratio duplicem determinationem requirat, series ita est fingenda, ut duabus conditionibus ex natura rei petitis satisfaciat. Verum haec methodus etiam ad alias investigationes extenditur, quae adeo in quantitatibus algebraicis versantur, a cuiusmodi exemplo hic inchoemus.

PROBLEMA 15

179. *Hanc expressionem* $y = \left(x + \sqrt{(1+xx)}\right)^n$ *in seriem secundum potestates ipsius* x *progredientem convertere.*

SOLUTIO

Quia est $ly = nl\left(x + \sqrt{(1+xx)}\right)$, erit

$$\frac{dy}{y} = \frac{ndx}{\sqrt{(1+xx)}};$$

iam ad signum radicale tollendum sumantur quadrata; erit

$$(1+xx)dy^2 = nnyydx^2.$$

Aequatio sumto dx constante denuo differentietur, ut per $2dy$ diviso prodeat

$$ddy(1+xx) + xdxdy - nnydx^2 = 0,$$

1) *Introductio in analysin infinitorum*, t. I cap. VII; vide etiam notam p. 76. L. S.

unde y per seriem elici debet. Primo autem patet, si sit $x=0$, fore $y=1$ ac, si x infinite parvum, $y=(1+x)^n=1+nx$. Fingatur ergo talis series

$$y=1+nx+Ax^2+Bx^3+Cx^4+Dx^5+Ex^6+\text{etc.},$$

ex qua colligitur

$$\frac{dy}{dx}=n+2Ax+3Bxx+4Cx^3+5Dx^4+6Ex^5+\text{etc.}$$

et

$$\frac{ddy}{dx^2}=2A+6Bx+12Cxx+20Dx^3+30Ex^4+\text{etc.}$$

Facta ergo substitutione adipiscimur

$$\left.\begin{array}{l}2A+6Bx+12Cxx+20Dx^3+30Ex^4+42Fx^5+\text{etc.}\\ \qquad\qquad +\ 2A\ \ +\ 6B\ \ +12C\ \ +20D\\ +\ \ n\ +\ 2A\ \ +\ 3B\ \ +\ 4C\ \ +\ 5D\\ -nn-\ n^3\ -An^2\ -Bn^2\ -Cn^2\ -Dn^2\end{array}\right\}=0$$

hincque derivantur sequentes determinationes

$$A=\frac{nn}{2},\quad B=\frac{n(nn-1)}{2\cdot 3},\quad C=\frac{A(nn-4)}{3\cdot 4},\quad D=\frac{B(nn-9)}{4\cdot 5}\quad \text{etc.},$$

ita ut sit

$$y=1+nx+\frac{nn}{1\cdot 2}x^2+\frac{n(nn-1)}{1\cdot 2\cdot 3}x^3+\frac{nn(nn-4)}{1\cdot 2\cdot 3\cdot 4}x^4+\frac{n(nn-1)(nn-9)}{1\cdot 2\cdot 3\cdot 4\cdot 5}x^5$$
$$+\frac{nn(nn-4)(nn-16)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}x^6+\frac{n(nn-1)(nn-9)(nn-25)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}x^7+\text{etc.}$$

COROLLARIUM 1

180. Uti est $y=\left(x+\sqrt{(1+xx)}\right)^n$, si statuamus $z=\left(-x+\sqrt{(1+xx)}\right)^n$, pro z similis series prodit, in qua x tantum negative capitur; hinc ergo concluditur

$$\frac{y+z}{2}=1+\frac{nn}{1\cdot 2}x^2+\frac{nn(nn-4)}{1\cdot 2\cdot 3\cdot 4}x^4+\frac{nn(nn-4)(nn-16)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}x^6+\text{etc.}$$

et

$$\frac{y-z}{2}=nx+\frac{n(nn-1)}{1\cdot 2\cdot 3}x^3+\frac{n(nn-1)(nn-9)}{1\cdot 2\cdot 3\cdot 4\cdot 5}x^5+\frac{n(nn-1)(nn-9)(nn-25)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}x^7+\text{etc.}$$

COROLLARIUM 2

181. Si ponatur $x = \sqrt{-1} \cdot \sin.\varphi$, erit $\sqrt{(1+xx)} = \cos.\varphi$ hincque

$$y = (\cos.\varphi + \sqrt{-1} \cdot \sin.\varphi)^n = \cos.n\varphi + \sqrt{-1} \cdot \sin.n\varphi$$

et

$$z = (\cos.\varphi - \sqrt{-1} \cdot \sin.\varphi)^n = \cos.n\varphi - \sqrt{-1} \cdot \sin.n\varphi,$$

unde deducimus

$$\cos.n\varphi = 1 - \frac{nn}{1\cdot 2}\sin.\varphi^2 + \frac{nn(nn-4)}{1\cdot 2\cdot 3\cdot 4}\sin.\varphi^4 - \frac{nn(nn-4)(nn-16)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\sin.\varphi^6 + \text{etc.},$$

$$\sin.n\varphi = n\sin.\varphi - \frac{n(nn-1)}{1\cdot 2\cdot 3}\sin.\varphi^3 + \frac{n(nn-1)(nn-9)}{1\cdot 2\cdot 3\cdot 4\cdot 5}\sin.\varphi^5$$
$$- \frac{n(nn-1)(nn-9)(nn-25)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}\sin.\varphi^7 + \text{etc.}$$

COROLLARIUM 3

182. Hae series ad multiplicationem angulorum pertinent atque hoc habent singulare, quod prior tantum casibus, quibus n est numerus par, posterior vero, quibus est numerus impar, abrumpatur.

PROBLEMA 16

183. *Proposito angulo φ tam eius sinum quam cosinum per seriem infinitam exprimere.*

SOLUTIO

Sit $y = \sin.\varphi$ et $z = \cos.\varphi$; erit

$$dy = d\varphi\sqrt{(1-yy)} \quad \text{et} \quad dz = -d\varphi\sqrt{(1-zz)}.$$

Sumantur quadrata

$$dy^2 = d\varphi^2(1-yy) \quad \text{et} \quad dz^2 = d\varphi^2(1-zz);$$

differentietur sumto $d\varphi$ constante fietque

$$ddy = -y\,d\varphi^2 \quad \text{et} \quad ddz = -z\,d\varphi^2$$

sicque y et z ex eadem aequatione definiri oportet. Sed pro $y = \sin.\varphi$ observandum est, si φ evanescat, fieri $y = \varphi$, pro $z = \cos.\varphi$, si φ evanescat, fieri $z = 1 - \frac{1}{2}\varphi\varphi$ seu $z = 1 + 0\varphi$. Fingatur ergo

$$y = \varphi + A\varphi^3 + B\varphi^5 + C\varphi^7 + \text{etc.},$$
$$z = 1 + \alpha\varphi^2 + \beta\varphi^4 + \gamma\varphi^6 + \text{etc.}$$

fietque substitutione facta

$$\left.\begin{array}{l} 2\cdot 3A\varphi + 4\cdot 5B\varphi^3 + 6\cdot 7C\varphi^5 + \text{etc.} \\ +1 \quad + \quad A \quad + \quad B \end{array}\right\} = 0$$

et

$$\left.\begin{array}{l} 1\cdot 2\alpha + 3\cdot 4\beta\varphi^2 + 5\cdot 6\gamma\varphi^4 + \text{etc.} \\ +1 + \quad \alpha \quad + \quad \beta \end{array}\right\} = 0,$$

unde colligimus

$$A = \frac{-1}{2\cdot 3}, \quad B = \frac{-A}{4\cdot 5}, \quad C = \frac{-B}{6\cdot 7}, \quad D = \frac{-C}{8\cdot 9} \quad \text{etc.},$$

$$\alpha = \frac{-1}{1\cdot 2}, \quad \beta = \frac{-\alpha}{3\cdot 4}, \quad \gamma = \frac{-\beta}{5\cdot 6}, \quad \delta = \frac{-\gamma}{7\cdot 8} \quad \text{etc.},$$

unde series iam notissimae obtinentur

$$\sin.\varphi = \frac{\varphi}{1} - \frac{\varphi^3}{1\cdot 2\cdot 3} + \frac{\varphi^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{\varphi^7}{1\cdot 2\cdots 7} + \text{etc.},$$

$$\cos.\varphi = 1 - \frac{\varphi^2}{1\cdot 2} + \frac{\varphi^4}{1\cdot 2\cdot 3\cdot 4} - \frac{\varphi^6}{1\cdot 2\cdots 6} + \text{etc.}$$

SCHOLION

184. Non opus erat ad differentialia secundi gradus descendere, sed ex formularum $y = \sin.\varphi$ et $z = \cos.\varphi$ differentialibus, quae sunt $dy = zd\varphi$ et $dz = -yd\varphi$, eaedem series facile reperiuntur. Fictis enim seriebus ut ante

$$y = \varphi + A\varphi^3 + B\varphi^5 + C\varphi^7 + \text{etc.} \quad \text{et} \quad z = 1 + \alpha\varphi^2 + \beta\varphi^4 + \gamma\varphi^6 + \text{etc.}$$

substitutione facta obtinebitur ex priore

$$\left.\begin{array}{l} 1 + 3A\varphi^2 + 5B\varphi^4 + 7C\varphi^6 + \text{etc.} \\ -1 - \quad \alpha \quad - \quad \beta \quad - \quad \gamma \end{array}\right\} = 0,$$

ex posteriore

$$\left.\begin{array}{l}2\alpha\varphi + 4\beta\varphi^3 + 6\gamma\varphi^5 + \text{etc.}\\ +1 \quad + A \quad\; + B\end{array}\right\} = 0,$$

unde colliguntur hae determinationes

$$\alpha = -\frac{1}{2}, \quad A = \frac{\alpha}{3}, \quad \beta = \frac{-A}{4}, \quad B = \frac{\beta}{5}, \quad \gamma = \frac{-B}{6}, \quad C = \frac{\gamma}{7} \quad \text{etc.}$$

ideoque

$$\alpha = -\frac{1}{2}, \quad \beta = +\frac{1}{2\cdot 3\cdot 4}, \quad \gamma = -\frac{1}{2\cdot 3\cdot 4\cdot 5\cdot 6} \quad \text{etc.},$$

$$A = -\frac{1}{2\cdot 3}, \quad B = +\frac{1}{2\cdot 3\cdot 4\cdot 5}, \quad C = -\frac{1}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} \quad \text{etc.},$$

qui valores cum praecedentibus conveniunt. Hinc intelligitur, quomodo saepe duae aequationes simul facilius per series evolvuntur, quam si alteram seorsim tractare velimus.

PROBLEMA 17

185. *Per seriem exprimere valorem quantitatis y, qui satisfaciat huic aequationi*

$$\frac{m\,dy}{\sqrt{(a+byy)}} = \frac{n\,dx}{\sqrt{(f+gxx)}}.$$

SOLUTIO

Integratio huius aequationis suppeditat

$$\frac{m}{\sqrt{b}}\, l\left(\sqrt{(a+byy)} + y\sqrt{b}\right) = \frac{n}{\sqrt{g}}\, l\left(\sqrt{(f+gxx)} + x\sqrt{g}\right) + C,$$

unde deducimus

$$y = \frac{1}{2\sqrt{b}}\left(\frac{\sqrt{(f+gxx)}+x\sqrt{g}}{h}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}} - \frac{a}{2\sqrt{b}}\left(\frac{\sqrt{(f+gxx)}-x\sqrt{g}}{k}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}$$

constantes h et k ita capiendo, ut sit $hk = f$. Hinc discimus, si x sumatur evanescens, fore

$$y = \frac{1}{2\sqrt{b}}\left(\frac{\sqrt{f}+x\sqrt{g}}{h}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}} - \frac{a}{2\sqrt{b}}\left(\frac{\sqrt{f}-x\sqrt{g}}{k}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}$$

seu

$$y = \frac{1}{2\sqrt{b}}\left(\left(\frac{\sqrt{k}}{\sqrt{h}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}} - a\left(\frac{\sqrt{h}}{\sqrt{k}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}\right) + \frac{nx}{2m\sqrt{f}}\left(\left(\frac{\sqrt{k}}{\sqrt{h}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}} + a\left(\frac{\sqrt{h}}{\sqrt{k}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}\right)$$

vel posito $y = A + Bx$ erit

$$B = \frac{n\sqrt{(AAb+a)}}{m\sqrt{f}},$$

ita ut constans B definiatur ex constante

$$A = \frac{1}{2\sqrt{b}}\left(\left(\frac{\sqrt{k}}{\sqrt{h}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}} - a\left(\frac{\sqrt{h}}{\sqrt{k}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}\right)$$

et vicissim

$$\left(\frac{\sqrt{k}}{\sqrt{h}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}} = A\sqrt{b} + \sqrt{(a+bAA)} \quad \text{atque} \quad a\left(\frac{\sqrt{h}}{\sqrt{k}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}} = -A\sqrt{b} + \sqrt{(a+bAA)}.$$

Nunc ad seriem inveniendam aequatio proposita sumtis quadratis

$$mm(f+gxx)dy^2 = nn(a+byy)dx^2$$

denuo differentietur capto dx constante, ut facta divisione per $2dy$ prodeat

$$mmddy(f+gxx) + mmgxdxdy - nnbydx^2 = 0.$$

Iam pro y fingatur series

$$y = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \text{etc.},$$

qua substituta habebitur

$$\left.\begin{array}{rrrrl} 2mmfC & +\,6mmfDx & +\,12mmfEx^2 & +\,20mmfFx^3 & +\text{ etc.} \\ & & +\,2mmgC & +\,6mmgD & \\ & +\,mmgB & +\,2mmgC & +\,3mmgD & \\ -\,nnbA & -\,nnbB & -\,nnbC & -\,nnbD & \end{array}\right\} = 0.$$

Cum ergo A et B dentur, reliquae litterae ita determinantur

$$C = \frac{nnb}{2mmf}A,$$

$$D = \frac{nnb - mmg}{2\cdot 3mmf}B, \quad E = \frac{nnb - 4mmg}{3\cdot 4mmf}C,$$

$$F = \frac{nnb - 9mmg}{4\cdot 5mmf}D, \quad G = \frac{nnb - 16mmg}{5\cdot 6mmf}E,$$

$$H = \frac{nnb - 25mmg}{6\cdot 7mmf}F, \quad I = \frac{nnb - 36mmg}{7\cdot 8mmf}G$$

sicque series pro y erit cognita.

EXEMPLUM 1

186. *Functionem transcendentem* $c^{\text{Arc. sin.}\, x}$ *per seriem secundum potestates ipsius* x *progredientem exprimere.*

Ponatur $y = c^{\text{Arc. sin.}\, x}$; erit $ly = lc\cdot \text{Arc. sin.}\, x$ et $\frac{dy}{y} = \frac{dxlc}{\sqrt{(1-xx)}}$, hinc

$$dy^2(1-xx) = yydx^2(lc)^2$$

et differentiando

$$ddy(1-xx) - xdxdy - ydx^2(lc)^2 = 0.$$

Observetur iam posito x evanescente fore $y = c^x = 1 + xlc$; hinc fingatur series

$$y = 1 + xlc + Ax^2 + Bx^3 + Cx^4 + Dx^5 + \text{etc.},$$

qua substituta habebitur

$$\left.\begin{array}{rrrrr} 1\cdot 2A & + 2\cdot 3Bx & + 3\cdot 4Cx^2 & + 4\cdot 5Dx^3 & + 5\cdot 6Ex^4 + \text{etc.} \\ & & -1\cdot 2A & -2\cdot 3B & -3\cdot 4C \\ & -\quad lc & -\quad 2A & -\quad 3B & -\quad 4C \\ -(lc)^2 & -\quad (lc)^3 & -A(lc)^2 & -B(lc)^2 & -C(lc)^2 \end{array}\right\} = 0,$$

unde reliqui coefficientes ita definiuntur

$$A = \frac{(lc)^2}{1\cdot 2}, \qquad C = \frac{4+(lc)^2}{3\cdot 4}A, \qquad E = \frac{16+(lc)^2}{5\cdot 6}C \quad \text{etc.},$$

$$B = \frac{(1+(lc)^2)lc}{2\cdot 3}, \qquad D = \frac{9+(lc)^2}{4\cdot 5}B, \qquad F = \frac{25+(lc)^2}{6\cdot 7}D \quad \text{etc.}$$

Sit brevitatis gratia $lc = \gamma$ eritque

$$c^{\text{Arc. sin.}\,x} = 1 + \gamma x + \frac{\gamma\gamma}{1\cdot 2}x^2 + \frac{\gamma(1+\gamma\gamma)}{1\cdot 2\cdot 3}x^3 + \frac{\gamma\gamma(4+\gamma\gamma)}{1\cdot 2\cdot 3\cdot 4}x^4$$
$$+ \frac{\gamma(1+\gamma\gamma)(9+\gamma\gamma)}{1\cdot 2\cdot 3\cdot 4\cdot 5}x^5 + \frac{\gamma\gamma(4+\gamma\gamma)(16+\gamma\gamma)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}x^6 + \text{etc.}$$

EXEMPLUM 2

187. *Posito $x = \sin.\varphi$ invenire seriem secundum potestates ipsius x progredientem, quae sinum anguli $n\varphi$ exprimat.*

Ponatur $y = \sin. n\varphi$ ac notetur evanescente φ fieri $x = \varphi$ et $y = n\varphi = nx$, hoc est $y = 0 + nx$, quod est seriei quaesitae initium. Nunc autem est

$$d\varphi = \frac{dx}{\sqrt{(1-xx)}} \quad \text{et} \quad nd\varphi = \frac{dy}{\sqrt{(1-yy)}}.$$

Ergo

$$\frac{dy}{\sqrt{(1-yy)}} = \frac{ndx}{\sqrt{(1-xx)}}$$

et sumtis quadratis

$$(1-xx)dy^2 = nndx^2(1-yy),$$

hinc

$$ddy(1-xx) - xdxdy + nnydx^2 = 0.$$

Quare fingatur haec series

$$y = nx + Ax^3 + Bx^5 + Cx^7 + Dx^9 + \text{etc.};$$

qua substituta habebitur

$$\left.\begin{array}{rrrr} 2\cdot 3Ax & +\, 4\cdot 5Bx^3 & +\, 6\cdot 7Cx^5 & +\, 8\cdot 9Dx^7 + \text{etc.} \\ & -\,2\cdot 3A & -\,4\cdot 5B & -\,6\cdot 7C \\ -\,n & -\,3A & -\,5B & -\,7C \\ +\,n^3 & +\,nnA & +\,nnB & +\,nnC \end{array}\right\} = 0,$$

unde hae determinationes colliguntur

$$A = \frac{-n(nn-1)}{2\cdot 3}, \quad B = \frac{-(nn-9)A}{4\cdot 5}, \quad C = \frac{-(nn-25)B}{6\cdot 7} \quad \text{etc.},$$

ita ut sit

$$y = nx - \frac{n(nn-1)}{1\cdot 2\cdot 3}x^3 + \frac{n(nn-1)(nn-9)}{1\cdot 2\cdot 3\cdot 4\cdot 5}x^5 - \frac{n(nn-1)(nn-9)(nn-25)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}x^7 + \text{etc.}$$

sive

$$\sin. n\varphi = n\sin.\varphi - \frac{n(nn-1)}{1\cdot 2\cdot 3}\sin.\varphi^3 + \frac{n(nn-1)(nn-9)}{1\cdot 2\cdot 3\cdot 4\cdot 5}\sin.\varphi^5 - \text{etc.}$$

SCHOLION

188. Quia haec series tantum casibus, quibus n est numerus impar, abrumpitur, pro paribus notandum est seriem commode exprimi posse per productum ex $\sin.\varphi$ in aliam seriem secundum cosinus ipsius φ potestates progredientem. Ad quam inveniendam ponamus $\cos.\varphi = u$ sitque

$$\sin. n\varphi = z\sin.\varphi = z\sqrt{(1-uu)},$$

unde ob

$$d\varphi = -\frac{du}{\sqrt{(1-uu)}}$$

erit differentiando

$$-\frac{ndu\cos. n\varphi}{\sqrt{(1-uu)}} = dz\sqrt{(1-uu)} - \frac{zudu}{\sqrt{(1-uu)}}$$

seu

$$-ndu\cos. n\varphi = dz(1-uu) - zudu,$$

quae sumto du constante denuo differentiata dat

$$-\frac{nndu^2\sin. n\varphi}{\sqrt{(1-uu)}} = ddz(1-uu) - 3ududz - zdu^2 = -nnzdu^2$$

ob $\frac{\sin. n\varphi}{\sqrt{(1-uu)}} = z$.

Quocirca series quaesita pro $z = \frac{\sin. n\varphi}{\sin.\varphi}$ ex hac aequatione erui debet

$$ddz(1-uu) - 3ududz - zdu^2 + nnzdu^2 = 0,$$

ubi notandum est, quia $u = \cos.\varphi$, evanescente u, quo casu fit $\varphi = 90^0$, fore vel $z = 0$, si n numerus par, vel $z = 1$, si $n = 4\alpha + 1$, vel $z = -1$, si $n = 4\alpha - 1$. Qui singuli casus seorsim sunt evolvendi; et quo principium cuiusque seriei pateat, sit $\varphi = 90^0 - \omega$ et evanescente ω fit $u = \cos.\varphi = \omega$, $\sin.\varphi = 1$, $\sin.n\varphi = \sin.(90^0 \cdot n - n\omega) = z$. Nunc pro casibus singulis

$$\begin{array}{llll} \text{I.} & \text{si } n = 4\alpha, & \text{fit } z = -\sin.n\omega = -nu \\ \text{II.} & \text{si } n = 4\alpha + 1, & \text{fit } z = \cos.n\omega = 1 \\ \text{III.} & \text{si } n = 4\alpha + 2, & \text{fit } z = \sin.n\omega = +nu \\ \text{IV.} & \text{si } n = 4\alpha + 3, & \text{fit } z = -\cos.n\omega = -1, \end{array}$$

unde series iam satis notae deducuntur.

CAPUT IV

DE INTEGRATIONE FORMULARUM LOGARITHMICARUM ET EXPONENTIALIUM

PROBLEMA 18

189. *Si X designet functionem algebraicam ipsius x, invenire integrale formulae $Xdxlx$.*

SOLUTIO

Quaeratur integrale $\int Xdx$, quod sit $=Z$, et cum quantitatis Zlx differentiale sit $dZlx+\frac{Zdx}{x}$, erit

$$Zlx=\int dZlx+\int\frac{Zdx}{x}$$

ideoque

$$\int dZlx=\int Xdxlx=Zlx-\int\frac{Zdx}{x}.$$

Sicque integratio formulae propositae reducta est ad integrationem huius $\frac{Zdx}{x}$, quae, si Z fuerit functio algebraica ipsius x, non amplius logarithmum involvit ideoque per praecedentes regulas tractari poterit. Sin autem $\int Xdx$ algebraice exhiberi nequeat, hinc nihil subsidii nascitur expedietque indicatione integralis $\int Xdxlx$ acquiescere eiusque valorem per approximationem investigare.

Nisi forte sit $X=\frac{1}{x}$, quo casu manifesto dat

$$\int\frac{dx}{x}lx=\frac{1}{2}(lx)^2+C.$$

COROLLARIUM 1

190. Eodem modo, si denotante V functionem quamcunque ipsius x proposita sit formula $XdxlV$, erit existente $\int Xdx = Z$ eius integrale $= ZlV - \int \frac{ZdV}{V}$ sicque ad formulam algebraicam reducitur, si modo Z algebraice detur.

COROLLARIUM 2

191. Pro casu singulari $\frac{dx}{x}lx$ notare licet, si posito $lx = u$ fuerit U functio quaecunque algebraica ipsius u, integrationem huius formulae $\frac{Udx}{x}$ non fore difficilem, quia ob $\frac{dx}{x} = du$ abit in Udu, cuius integratio ad praecedentia capita refertur.

SCHOLION

192. Haec reductio innititur isti fundamento, quod, cum sit

$$d.xy = ydx + xdy,$$

hinc vicissim fiat $xy = \int ydx + \int xdy$ ideoque

$$\int ydx = xy - \int xdy,$$

ita ut hoc modo in genere integratio formulae ydx ad integrationem formulae xdy reducatur. Quodsi ergo proposita quacunque formula Vdx functio V in duos factores, puta $V = PQ$, resolvi queat, ita ut integrale $\int Pdx = S$ assignari queat, ob $Pdx = dS$ erit $Vdx = PQdx = QdS$ hincque

$$\int Vdx = QS - \int SdQ.$$

Huiusmodi reductio insignem usum affert, cum formula $\int SdQ$ simplicior fuerit quam proposita $\int Vdx$ eaque insuper simili modo ad simpliciorem reduci queat. Interdum etiam commode evenit, ut hac methodo tandem ad formulam propositae similem perveniatur, quo casu integratio pariter obtinetur. Veluti si ulteriori reductione inveniremus $\int SdQ = T + n\int Vdx$, foret utique $\int Vdx = QS - T - n\int Vdx$ hincque

$$\int Vdx = \frac{QS - T}{n+1}.$$

Tum igitur talis reductio insignem praestat usum, cum vel ad formulam simpliciorem vel ad eandem perducit. Atque ex hoc principio praecipuos casus, quibus formula $Xdxlx$ vel integrationem admittit vel per seriem commode exhiberi potest, evolvamus.

EXEMPLUM 1

193. *Formulae differentialis $x^n dxlx$ integrale invenire denotante n numerum quemcunque.*

Cum sit $\int x^n dx = \frac{1}{n+1} x^{n+1}$, erit

$$\int x^n dxlx = \frac{1}{n+1} x^{n+1} lx - \int \frac{1}{n+1} x^{n+1} d.lx$$

$$= \frac{1}{n+1} x^{n+1} lx - \frac{1}{n+1} \int x^n dx = \frac{1}{n+1} x^{n+1} lx - \frac{1}{(n+1)^2} x^{n+1}$$

ideoque

$$\int x^n dxlx = \frac{1}{n+1} x^{n+1} \left(lx - \frac{1}{n+1} \right).$$

Sicque haec formula absolute est integrabilis.

COROLLARIUM 1

194. Casus simpliciores, quibus n est numerus integer sive positivus sive negativus, tenuisse iuvabit

$$\int dxlx = xlx - x, \qquad \int \frac{dx}{xx} lx = -\frac{1}{x} lx - \frac{1}{x},$$

$$\int xdxlx = \frac{1}{2} xxlx - \frac{1}{4} xx, \qquad \int \frac{dx}{x^3} lx = -\frac{1}{2xx} lx - \frac{1}{4xx},$$

$$\int x^2 dxlx = \frac{1}{3} x^3 lx - \frac{1}{9} x^3, \qquad \int \frac{dx}{x^4} lx = -\frac{1}{3x^3} lx - \frac{1}{9x^3},$$

$$\int x^3 dxlx = \frac{1}{4} x^4 lx - \frac{1}{16} x^4, \qquad \int \frac{dx}{x^5} lx = -\frac{1}{4x^4} lx - \frac{1}{16x^4}.$$

COROLLARIUM 2

195. Casum $\int \frac{dx}{x} lx = \frac{1}{2}(lx)^2$, qui est omnino singularis, iam supra annotavimus, sequitur vero etiam ex reductione ad eandem formulam. Namque

per superiorem reductionem habemus

$$\int \frac{dx}{x} lx = lx \cdot lx - \int lx \cdot d.lx = (lx)^2 - \int \frac{dx}{x} lx$$

hincque $2\int \frac{dx}{x} lx = (lx)^2$, consequenter

$$\int \frac{dx}{x} lx = \frac{1}{2}(lx)^2.$$

EXEMPLUM 2

196. *Formulae* $\frac{dx}{1-x} lx$ *integrale per seriem exprimere.*

Reductione ante adhibita parum lucramur; prodit enim

$$\int \frac{dx}{1-x} lx = l\frac{1}{1-x} \cdot lx - \int \frac{dx}{x} l\frac{1}{1-x}.$$

Cum autem sit

$$l\frac{1}{1-x} = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \text{etc.},$$

erit

$$\int \frac{dx}{x} l\frac{1}{1-x} = x + \frac{1}{4}x^2 + \frac{1}{9}x^3 + \frac{1}{16}x^4 + \frac{1}{25}x^5 + \text{etc.}$$

ideoque

$$\int \frac{dx}{1-x} lx = l\frac{1}{1-x} \cdot lx - x - \frac{1}{4}x^2 - \frac{1}{9}x^3 - \frac{1}{16}x^4 - \frac{1}{25}x^5 - \text{etc.},$$

quod integrale evanescit casu $x = 0$; etsi enim lx tum in infinitum abit, tamen $l\frac{1}{1-x} = x + \frac{1}{2}x^2 + \frac{1}{3}x^3$ etc. ita evanescit, ut, etiam si per lx multiplicetur, in nihilum abeat; est enim in genere $x^n lx = 0$ posito $x = 0$, dum n numerus positivus.

COROLLARIUM 1

197. Si ponamus $1 - x = u$, fit

$$\frac{dx}{1-x} lx = -\frac{du}{u} l(1-u) = \frac{du}{u} l\frac{1}{1-u}$$

ideoque

$$\int \frac{dx}{1-x} lx = C + u + \frac{1}{4}u^2 + \frac{1}{9}u^3 + \frac{1}{16}u^4 + \frac{1}{25}u^5 + \text{etc.};$$

quae ut etiam casu $x=0$ seu $u=1$ evanescat, capi debet

$$C=-1-\frac{1}{4}-\frac{1}{9}-\frac{1}{16}-\frac{1}{25}-\text{etc.}=-\frac{1}{6}\pi\pi.$$

COROLLARIUM 2

198. Sumto ergo $1-x=u$ seu $x+u=1$ aequales erunt inter se hae expressiones

$$-lx\cdot lu-x-\frac{1}{4}x^2-\frac{1}{9}x^3-\frac{1}{16}x^4-\text{etc.}$$

$$=-\frac{1}{6}\pi^2+u+\frac{1}{4}u^2+\frac{1}{9}u^3+\frac{1}{16}u^4+\text{etc.}$$

seu erit

$$\frac{1}{6}\pi^2-lx\cdot lu=x+u+\frac{1}{4}(x^2+u^2)+\frac{1}{9}(x^3+u^3)+\frac{1}{16}(x^4+u^4)+\text{etc.}$$

COROLLARIUM 3

199. Haec series maxime convergit ponendo $x=u=\frac{1}{2}$; hoc ergo casu habebimus

$$\frac{1}{6}\pi^2-(l2)^2=1+\frac{1}{2\cdot 4}+\frac{1}{4\cdot 9}+\frac{1}{8\cdot 16}+\frac{1}{16\cdot 25}+\frac{1}{32\cdot 36}+\text{etc.}$$

Huius ergo seriei

$$x+\frac{1}{4}x^2+\frac{1}{9}x^3+\frac{1}{16}x^4+\frac{1}{25}x^5+\text{etc.}$$

summa habetur non solum casu $x=1$, quo est $=\frac{\pi\pi}{6}$, sed etiam casu $x=\frac{1}{2}$, quo est $=\frac{1}{12}\pi^2-\frac{1}{2}(l2)^2$.

COROLLARIUM 4

200. Si ponamus $x=\frac{1}{3}$ et $u=\frac{2}{3}$, erit huius seriei

$$1+\frac{5}{3^2\cdot 4}+\frac{9}{3^3\cdot 9}+\frac{17}{3^4\cdot 16}+\frac{33}{3^5\cdot 25}+\frac{65}{3^6\cdot 36}+\text{etc.},$$

cuius terminus generalis $= \frac{1+2^n}{3^n nn}$, summa $= \frac{1}{6}\pi^2 - l3 \cdot l\frac{3}{2}$, neque vero hinc seriei

$$x + \frac{1}{4}x^2 + \frac{1}{9}x^3 + \frac{1}{16}x^4 + \text{etc.}$$

binos casus $x = \frac{1}{3}$ et $x = \frac{2}{3}$ seorsim summare licet.

EXEMPLUM 3

201. *Formulae* $\frac{dx}{(1-x)^2}lx$ *integrale invenire idemque in seriem convertere.*

Cum sit $\int \frac{dx}{(1-x)^2} = \frac{1}{1-x}$, erit

$$\int \frac{dx}{(1-x)^2}lx = \frac{1}{1-x}lx - \int \frac{dx}{x(1-x)},$$

at ob $\frac{1}{x(1-x)} = \frac{1}{x} + \frac{1}{1-x}$ fit

$$\int \frac{dx}{x(1-x)} = lx + l\frac{1}{1-x},$$

unde colligimus integrale

$$\int \frac{dx}{(1-x)^2}lx = \frac{lx}{1-x} - lx - l\frac{1}{1-x} = \frac{xlx}{1-x} - l\frac{1}{1-x}$$

ita sumtum, ut evanescat posito $x = 0$.

Iam pro serie commodissime invenienda statuatur $1 - x = u$ et nostra formula fit

$$= \frac{-du}{uu}l(1-u) = \frac{du}{uu}l\frac{1}{1-u} = \frac{du}{uu}\left(u + \frac{1}{2}u^2 + \frac{1}{3}u^3 + \frac{1}{4}u^4 + \frac{1}{5}u^5 + \text{etc.}\right).$$

Quocirca integrando nanciscimur

$$\int \frac{dx}{(1-x)^2}lx = C + lu + \frac{u}{1\cdot 2} + \frac{uu}{2\cdot 3} + \frac{u^3}{3\cdot 4} + \frac{u^4}{4\cdot 5} + \text{etc.};$$

quae expressio ut etiam evanescat facto $x = 0$ seu $u = 1$, oportet sit

$$C = -\frac{1}{1\cdot 2} - \frac{1}{2\cdot 3} - \frac{1}{3\cdot 4} - \frac{1}{4\cdot 5} - \text{etc.} = -1.$$

Quare ob $x = 1 - u$ obtinebimus

$$\frac{u}{1\cdot 2} + \frac{u^2}{2\cdot 3} + \frac{u^3}{3\cdot 4} + \frac{u^4}{4\cdot 5} + \text{etc.} = 1 - lu + \frac{(1-u)l(1-u)}{u} + lu$$
$$= 1 + \frac{(1-u)l(1-u)}{u}.$$

COROLLARIUM 1

202. Simili modo si $dy = \frac{du}{u\sqrt{u}}\, l\frac{1}{1-u}$, erit

$$y = -\frac{2}{\sqrt{u}}\, l\frac{1}{1-u} + \int \frac{2\,du}{(1-u)\sqrt{u}},$$

at posito $u = xx$ fit

$$\int \frac{2\,du}{(1-u)\sqrt{u}} = 4\int \frac{dx}{1-xx} = 2l\frac{1+x}{1-x}.$$

Ergo

$$y = 2l\frac{1+\sqrt{u}}{1-\sqrt{u}} - \frac{2}{\sqrt{u}}\, l\frac{1}{1-u}.$$

At quia per seriem

$$dy = \frac{du}{u\sqrt{u}}\left(u + \frac{1}{2}uu + \frac{1}{3}u^3 + \frac{1}{4}u^4 + \text{etc.}\right),$$

erit etiam

$$y = 2\sqrt{u} + \frac{2}{2\cdot 3}u\sqrt{u} + \frac{2}{3\cdot 5}u^2\sqrt{u} + \frac{2}{4\cdot 7}u^3\sqrt{u} + \text{etc.}$$

COROLLARIUM 2

203. Si ergo multiplicemus per $\frac{\sqrt{u}}{2}$, adipiscimur

$$u + \frac{uu}{2\cdot 3} + \frac{u^3}{3\cdot 5} + \frac{u^4}{4\cdot 7} + \frac{u^5}{5\cdot 9} + \text{etc.} = \sqrt{u}\cdot l\frac{1+\sqrt{u}}{1-\sqrt{u}} + l(1-u),$$

quae summa est etiam

$$= (1+\sqrt{u})\,l(1+\sqrt{u}) + (1-\sqrt{u})\,l(1-\sqrt{u}).$$

Quare sumto $u = 1$ ob $(1-\sqrt{u})\,l(1-\sqrt{u}) = 0$ erit

$$1 + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{4\cdot 7} + \frac{1}{5\cdot 9} + \frac{1}{6\cdot 11} + \text{etc.} = 2l2.$$

PROBLEMA 19

204. *Si P denotet functionem ipsius x, invenire integrale huius formulae* $dy = dP(lx)^n$.

SOLUTIO

Per reductionem supra monstratam fit

$$y = P(lx)^n - \int P d.(lx)^n = P(lx)^n - n\int \frac{P dx}{x}(lx)^{n-1}.$$

Hinc, si sit $\int \frac{P dx}{x} = Q$, erit simili modo

$$\int \frac{P dx}{x}(lx)^{n-1} = Q(lx)^{n-1} - (n-1)\int \frac{Q dx}{x}(lx)^{n-2}.$$

Quo modo si ulterius progredimur haecque integralia capere liceat

$$\int \frac{P dx}{x} = Q, \quad \int \frac{Q dx}{x} = R, \quad \int \frac{R dx}{x} = S, \quad \int \frac{S dx}{x} = T \text{ etc.},$$

obtinebimus integrale quaesitum

$$\int dP(lx)^n = P(lx)^n - nQ(lx)^{n-1} + n(n-1)R(lx)^{n-2} - n(n-1)(n-2)S(lx)^{n-3} + \text{etc.},$$

ac si exponens n fuerit numerus integer positivus, integrale forma finita exprimetur.

EXEMPLUM 1

205. *Formulae $x^m dx(lx)^2$ integrale assignare.*

Hic est $n = 2$ et $P = \frac{x^{m+1}}{m+1}$, hinc

$$Q = \frac{x^{m+1}}{(m+1)^2} \quad \text{et} \quad R = \frac{x^{m+1}}{(m+1)^3},$$

unde colligimus

$$\int x^m dx(lx)^2 = x^{m+1}\left(\frac{(lx)^2}{m+1} - \frac{2lx}{(m+1)^2} + \frac{2\cdot 1}{(m+1)^3}\right),$$

quod integrale evanescit posito $x = 0$, dum sit $m + 1 > 0$.

COROLLARIUM 1

206. Hinc posito $x=1$ fit $\int x^m dx(lx)^2=\frac{2\cdot 1}{(m+1)^3}$. Ex praecedentibus autem patet, si formula $\int x^m dx\,lx$ ita integretur, ut evanescat posito $x=0$, tum facto $x=1$ fieri $\int x^m dx\,lx=\frac{-1}{(m+1)^2}$.

COROLLARIUM 2

207. At si sit $m=-1$, ut habeatur $\frac{dx}{x}(lx)^2$, erit eius integrale

$$\int\frac{dx}{x}(lx)^2=\frac{1}{3}(lx)^3,$$

qui solus casus ex formula generali est excipiendus.

EXEMPLUM 2

208. *Formulae* $x^{m-1}dx(lx)^3$ *integrale assignare.*

Hic est $n=3$ et $P=\frac{x^m}{m}$, hinc

$$Q=\frac{x^m}{m^2},\quad R=\frac{x^m}{m^3}\quad \text{et}\quad S=\frac{x^m}{m^4},$$

unde integrale quaesitum fit

$$\int x^{m-1}dx(lx)^3=x^m\left(\frac{(lx)^3}{m}-\frac{3(lx)^2}{m^2}+\frac{3\cdot 2lx}{m^3}-\frac{3\cdot 2\cdot 1}{m^4}\right),$$

quod integrale evanescit posito $x=0$, dum sit $m>0$.

COROLLARIUM 1

209. Quodsi integrali ita sumto, ut evanescat posito $x=0$, tum ponatur $x=1$, erit

$$\int x^{m-1}dx=\frac{1}{m},\quad \int x^{m-1}dx\,lx=-\frac{1}{m^2},\quad \int x^{m-1}dx(lx)^2=+\frac{1\cdot 2}{m^3}$$

et

$$\int x^{m-1}dx(lx)^3=-\frac{1\cdot 2\cdot 3}{m^4}.$$

COROLLARIUM 2

210. Casu autem $m=0$ erit integrale $\int\frac{dx}{x}(lx)^3=\frac{1}{4}(lx)^4$, quod ita determinari nequit, ut evanescat posito $x=0$; oporteret enim constantem infinitam adiici. Hoc autem integrale evanescit posito $x=1$.

EXEMPLUM 3

211. *Formulae $x^{m-1}dx(lx)^n$ integrale assignare.*

Cum hic sit $P=\frac{x^m}{m}$, erit

$$Q=\frac{x^m}{m^2},\quad R=\frac{x^m}{m^3},\quad S=\frac{x^m}{m^4}\quad \text{etc.}$$

Hinc integrale quaesitum prodit

$$\int x^{m-1}dx(lx)^n=x^m\left(\frac{(lx)^n}{m}-\frac{n(lx)^{n-1}}{m^2}+\frac{n(n-1)(lx)^{n-2}}{m^3}-\frac{n(n-1)(n-2)(lx)^{n-3}}{m^4}+\text{etc.}\right).$$

Casu autem $m=0$ est

$$\int\frac{dx}{x}(lx)^n=\frac{1}{n+1}(lx)^{n+1}.$$

COROLLARIUM 1

212. Si $m>0$, integrale assignatum evanescit posito $x=0$; deinceps ergo si sumatur $x=1$, erit integrale

$$\int x^{m-1}dx(lx)^n=\pm\frac{1\cdot 2\cdot 3\cdots n}{m^{n+1}},$$

ubi signum $+$ valet, si n sit numerus par, inferius vero, si n impar.

COROLLARIUM 2

213. Haec ergo ambiguitas tollitur, si loco lx scribatur $-l\frac{1}{x}$; tum enim integratione eodem modo instituta positoque $x=1$ fiet

$$\int x^{m-1}dx\left(l\frac{1}{x}\right)^n=+\frac{1\cdot 2\cdot 3\cdots n}{m^{n+1}}.$$

SCHOLION

214. Si exponens n sit numerus fractus, integrale inventum per seriem infinitam exprimitur; veluti si sit $n = -\frac{1}{2}$, reperitur

$$\int \frac{x^{m-1}dx}{\sqrt{lx}} = x^m \left(\frac{1}{m\sqrt{lx}} + \frac{1}{2m^2(lx)^{\frac{3}{2}}} + \frac{1\cdot 3}{4m^3(lx)^{\frac{5}{2}}} + \frac{1\cdot 3\cdot 5}{8m^4(lx)^{\frac{7}{2}}} + \text{etc.} \right),$$

quae etiam, quatenus initio x ab 0 ad 1 crescere sumitur, hoc modo repraesentari potest

$$\int \frac{x^{m-1}dx}{\sqrt{l\frac{1}{x}}} = \frac{x^m}{\sqrt{l\frac{1}{x}}} \left(\frac{1}{m} + \frac{1}{2m^2 lx} + \frac{1\cdot 3}{4m^3(lx)^2} + \frac{1\cdot 3\cdot 5}{8m^4(lx)^3} + \text{etc.} \right).$$

Si exponens n sit negativus, etsi integer, tamen integrale inventum in infinitum progredietur; verum hoc casu alia ratione integrationem instituere licet, qua tandem reducitur ad huiusmodi formulam $\int \frac{Tdx}{lx}$, cuius integratio nullo modo simplicior reddi potest. Hanc ergo reductionem sequenti problemate doceamus.

PROBLEMA 20

215. *Integrationem huius formulae $dy = \frac{Xdx}{(lx)^n}$ continuo ad formulas simpliciores reducere.*

SOLUTIO

Formula proposita ita repraesentetur

$$dy = Xx \cdot \frac{dx}{x(lx)^n},$$

et cum sit $\int \frac{dx}{x(lx)^n} = \frac{-1}{(n-1)(lx)^{n-1}}$, erit

$$y = \frac{-Xx}{(n-1)(lx)^{n-1}} + \frac{1}{n-1} \int \frac{1}{(lx)^{n-1}} \cdot d(Xx).$$

Quare si ponamus continuo

$$d(Xx) = Pdx, \quad d(Px) = Qdx, \quad d(Qx) = Rdx \quad \text{etc.},$$

erit hanc reductionem continuando

$$y = \frac{-Xx}{(n-1)(lx)^{n-1}} - \frac{Px}{(n-1)(n-2)(lx)^{n-2}} - \frac{Qx}{(n-1)(n-2)(n-3)(lx)^{n-3}} - \text{etc.},$$

donec tandem perveniatur ad hanc integralem

$$+\frac{1}{(n-1)(n-2)\cdots 1}\int\frac{V\,dx}{lx},$$

ita ut, quoties n fuerit numerus integer positivus, integratio tandem ad huiusmodi formulam perducatur.

EXEMPLUM 1

216. *Formulae differentialis* $dy = \frac{x^{m-1}dx}{(lx)^2}$ *integrale investigare.*

Hic est $n = 2$ et $X = x^{m-1}$, unde fit $P = mx^{m-1}$, hincque integrale

$$y = \int\frac{x^{m-1}dx}{(lx)^2} = -\frac{x^m}{lx} + \frac{m}{1}\int\frac{x^{m-1}dx}{lx}.$$

At formulae $\frac{x^{m-1}dx}{lx}$ integrale exhiberi nequit, nisi casu $m = 0$, quo fit $\int\frac{dx}{xlx} = llx$. Verum si $m = 0$, formulae propositae integratio ne hinc quidem pendet; fit enim absolute

$$y = \int\frac{dx}{x(lx)^2} = -\frac{1}{lx} + C.$$

EXEMPLUM 2

217. *Formulae differentialis* $dy = \frac{x^{m-1}dx}{(lx)^n}$ *integrale investigare casibus, quibus* n *est numerus integer positivus.*

Cum sit $X = x^{m-1}$, erit

$$P = \frac{d.Xx}{dx} = mx^{m-1},$$

tum vero

$$Q = \frac{d.Px}{dx} = m^2x^{m-1}, \quad R = m^3x^{m-1}, \quad S = m^4x^{m-1} \quad \text{etc.}$$

Quare integrale hinc ita formabitur, ut sit

$$y = \int \frac{x^{m-1}dx}{(lx)^n} = -\frac{x^m}{(n-1)(lx)^{n-1}} - \frac{mx^m}{(n-1)(n-2)(lx)^{n-2}} - \frac{m^2x^m}{(n-1)(n-2)(n-3)(lx)^{n-3}}$$
$$- \cdots + \frac{m^{n-1}}{(n-1)(n-2)\cdots 1}\int \frac{x^{m-1}dx}{lx}.$$

COROLLARIUM

218. Pro n ergo successive numeros 1, 2, 3, 4 etc. substituendo habebimus istas reductiones

$$\int \frac{x^{m-1}dx}{(lx)^2} = -\frac{x^m}{lx} + \frac{m}{1}\int \frac{x^{m-1}dx}{lx},$$

$$\int \frac{x^{m-1}dx}{(lx)^3} = -\frac{x^m}{2(lx)^2} - \frac{mx^m}{2\cdot 1\, lx} + \frac{m^2}{2\cdot 1}\int \frac{x^{m-1}dx}{lx},$$

$$\int \frac{x^{m-1}dx}{(lx)^4} = -\frac{x^m}{3(lx)^3} - \frac{mx^m}{3\cdot 2(lx)^2} - \frac{m^2x^m}{3\cdot 2\cdot 1\, lx} + \frac{m^3}{3\cdot 2\cdot 1}\int \frac{x^{m-1}dx}{lx}.$$

SCHOLION

219.[1]) Hae ergo integrationes pendent a formula $\int \frac{x^{m-1}dx}{lx}$, quae posito $x^m = z$ ob $x^{m-1}dx = \frac{1}{m}dz$ et $lx = \frac{1}{m}lz$ reducitur ad hanc simplicissimam formam $\int \frac{dz}{lz}$; cuius integrale si assignari posset, amplissimum usum in Analysi esset allaturum, verum nullis adhuc artificiis neque per logarithmos neque angulos exhiberi potuit; quomodo autem per seriem exprimi possit, infra ostendemus (§ 228). Videtur ergo haec formula $\int \frac{dz}{lz}$ singularem speciem functionum transcendentium suppeditare, quae utique accuratiorem evolutionem meretur. Eadem autem quantitas transcendens in integrationibus formularum exponentialium frequenter occurrit, quas in hoc capite tractare instituimus, propterea quod cum logarithmicis tam arcte cohaerent, ut alterum genus facile in alterum converti possit; veluti ipsa formula modo

1) Cf. *Laurentii Mascheronii Adnotationes ad Calculum integralem Euleri*, Ticini 1790—1792; vide praecipue adnotationem I partis prioris nec non in altera parte adnotationem alteram ad cap. IV sect. I. Quas *Adnotationes* adiecimus volumini secundo *Institutionum Calculi integralis*; *Leonhardi Euleri Opera omnia*, series I, vol. 12. L. S.

considerata $\frac{dz}{lz}$ posito $lz = x$, ut sit $z = e^x$ et $dz = e^x dx$, transformatur in hanc exponentialem $e^x \frac{dx}{x}$, cuius ergo integratio aeque est abscondita. Formulas igitur tractabiles evolvamus et huiusmodi quidem, quae non obvia substitutione ad formam algebraicam reduci possunt. Veluti si V fuerit functio quaecunque ipsius v sitque $v = a^x$, formula Vdx ob $x = \frac{lv}{la}$ et $dx = \frac{dv}{vla}$ abit in $\frac{Vdv}{vla}$, quae ratione variabilis v est algebraica. Huiusmodi ergo formulas $\frac{a^x dx}{\sqrt{(1 + a^{nx})}}$, quippe quae posito $a^x = v$ nihil habent difficultatis, hinc excludimus.

PROBLEMA 21

220. *Formulae differentialis $a^x X dx$ denotante X functionem quamcunque ipsius x integrale investigare.*

SOLUTIO 1

Cum sit $d.a^x = a^x dx la$, erit vicissim $\int a^x dx = \frac{1}{la} \cdot a^x$; quare si formula proposita in hos factores resolvatur $X \cdot a^x dx$, habebitur per reductionem

$$\int a^x X dx = \frac{1}{la} a^x X - \frac{1}{la} \int a^x dX.$$

Quodsi ulterius ponamus $dX = Pdx$, ut sit

$$\int a^x P dx = \frac{1}{la} a^x P - \frac{1}{la} \int a^x dP,$$

prodibit haec reductio

$$\int a^x X dx = \frac{1}{la} a^x X - \frac{1}{(la)^2} a^x P + \frac{1}{(la)^2} \int a^x dP.$$

Si porro ponamus $dP = Qdx$, habebitur haec reductio

$$\int a_x X dx = \frac{1}{la} a^x X - \frac{1}{(la)^2} a^x P + \frac{1}{(la)^3} a^x Q - \frac{1}{(la)^3} \int a^x dQ$$

sicque ulterius ponendo $dQ = Rdx$, $dR = Sdx$ etc. progredi licet, donec ad formulam vel integrabilem vel in suo genere simplicissimam perveniatur.

SOLUTIO 2

Alio modo resolutio formulae in factores institui potest; ponatur

$$\int X dx = P \quad \text{seu} \quad X dx = dP$$

et formula ita relata $a^x \cdot dP$ habebitur

$$\int a^x X dx = a^x P - la \int a^x P dx;$$

simili modo si ponamus $\int P dx = Q$, obtinebimus

$$\int a^x X dx = a^x P - la \cdot a^x Q + (la)^2 \int a^x Q dx.$$

Ponamus porro $\int Q dx = R$ et consequimur

$$\int a^x X dx = a^x P - la \cdot a^x Q + (la)^2 \cdot a^x R - (la)^3 \int a^x R dx$$

hocque modo, quousque lubuerit, progredi licet, donec ad formulam vel integrabilem vel in suo genere simplicissimam perveniamus.

COROLLARIUM 1

221. Priori solutione semper uti licet, quia functiones P, Q, R etc. per differentiationem functionis X eliciuntur, dum est

$$P = \frac{dX}{dx}, \quad Q = \frac{dP}{dx}, \quad R = \frac{dQ}{dx} \text{ etc.}$$

Quare si X fuerit functio rationalis integra, tandem ad formulam pervenietur $\int a^x dx = \frac{1}{la} \cdot a^x$ ideoque his casibus integrale absolute exhiberi potest.

COROLLARIUM 2

222. Altera solutio locum non invenit, nisi formulae $X dx$ integrale P assignari queat; neque etiam eam continuare licet, nisi quatenus sequentes integrationes

$$\int P dx = Q, \quad \int Q dx = R \text{ etc.}$$

succedunt.

EXEMPLUM 1

223. *Formulae $a^x x^n dx$ integrale definire denotante n numerum integrum positivum.*

Cum sit $X = x^n$, solutione prima utentes habebimus

$$\int a^x x^n dx = \frac{1}{la} \cdot a^x x^n - \frac{n}{la} \int a^x x^{n-1} dx;$$

hinc ponendo pro n successive numeros 0, 1, 2, 3 etc., quia primo casu integratio constat, sequentia integralia eruemus

$$\int a^x dx = \frac{1}{la} a^x,$$

$$\int a^x x dx = \frac{1}{la} a^x x - \frac{1}{(la)^2} a^x,$$

$$\int a^x x^2 dx = \frac{1}{la} a^x x^2 - \frac{2}{(la)^2} a^x x + \frac{2 \cdot 1}{(la)^3} a^x,$$

$$\int a^x x^3 dx = \frac{1}{la} a^x x^3 - \frac{3}{(la)^2} a^x x^2 + \frac{3 \cdot 2}{(la)^3} a^x x - \frac{3 \cdot 2 \cdot 1}{(la)^4} a^x$$

etc.,

unde in genere pro quovis exponente n concludimus

$$\int a^x x^n dx = a^x \left(\frac{x^n}{la} - \frac{n x^{n-1}}{(la)^2} + \frac{n(n-1) x^{n-2}}{(la)^3} - \frac{n(n-1)(n-2) x^{n-3}}{(la)^4} + \text{etc.} \right),$$

ad quam expressionem insuper constantem arbitrariam adiici oportet, ut integrale completum obtineatur.

COROLLARIUM

224. Si integrale ita determinari debeat, ut evanescat posito $x = 0$, erit

$$\int a^x dx = \frac{1}{la} a^x - \frac{1}{la},$$

$$\int a^x x dx = a^x \left(\frac{x}{la} - \frac{1}{(la)^2} \right) + \frac{1}{(la)^2},$$

$$\int a^x x^2 dx = a^x \left(\frac{x^2}{la} - \frac{2x}{(la)^2} + \frac{2 \cdot 1}{(la)^3} \right) - \frac{2 \cdot 1}{(la)^3},$$

$$\int a^x x^3 dx = a^x \left(\frac{x^3}{la} - \frac{3x^2}{(la)^2} + \frac{3 \cdot 2x}{(la)^3} - \frac{3 \cdot 2 \cdot 1}{(la)^4} \right) + \frac{3 \cdot 2 \cdot 1}{(la)^4}$$

etc.

EXEMPLUM 2

225. *Formulae* $\frac{a^x dx}{x^n}$ *integrale investigare, si quidem n denotet numerum integrum positivum.*

Hic commode altera solutione utemur, ubi, cum sit $X = \frac{1}{x^n}$, erit

$$P = \frac{-1}{(n-1)x^{n-1}};$$

hincque resultat ista reductio

$$\int \frac{a^x dx}{x^n} = \frac{-a^x}{(n-1)x^{n-1}} + \frac{la}{n-1}\int \frac{a^x dx}{x^{n-1}}.$$

Perspicuum igitur est posito $n = 1$ hinc nihil concludi posse; qui est ipse casus supra memoratus $\int \frac{a^x dx}{x}$ singularem speciem transcendentium functionum complectens, qua admissa integralia sequentium casuum exhibere poterimus:

$$\int \frac{a^x dx}{x^2} = C - \frac{a^x}{1x} + \frac{la}{1}\int \frac{a^x dx}{x},$$

$$\int \frac{a^x dx}{x^3} = C - \frac{a^x}{2x^2} - \frac{a^x la}{2\cdot 1x} + \frac{(la)^2}{2\cdot 1}\int \frac{a^x dx}{x},$$

$$\int \frac{a^x dx}{x^4} = C - \frac{a^x}{3x^3} - \frac{a^x la}{3\cdot 2x^2} - \frac{a^x(la)^2}{3\cdot 2\cdot 1x} + \frac{(la)^3}{3\cdot 2\cdot 1}\int \frac{a^x dx}{x},$$

unde in genere colligimus

$$\int \frac{a^x dx}{x^n} = C - \frac{a^x}{(n-1)x^{n-1}} - \frac{a^x la}{(n-1)(n-2)x^{n-2}} - \frac{a^x(la)^2}{(n-1)(n-2)(n-3)x^{n-3}}$$
$$- \cdots - \frac{a^x(la)^{n-2}}{(n-1)(n-2)\cdots 1x} + \frac{(la)^{n-1}}{(n-1)(n-2)\cdots 1}\int \frac{a^x dx}{x}.$$

COROLLARIUM 1

226. Admissa ergo quantitate transcendente $\int \frac{a^x dx}{x}$ hanc formulam $a^x x^m dx$ integrare poterimus, sive exponens m fuerit numerus integer positivus sive negativus. Illis quidem casibus integratio ab ista nova quantitate transcendente non pendet.

COROLLARIUM 2

227. At si m fuerit fractus numerus, neutra solutio negotium conficit, sed utraque seriem infinitam pro integrali exhibet. Veluti si sit $m = -\frac{1}{2}$, habebimus ex priore

$$\int \frac{a^x dx}{\sqrt{x}} = a^x\left(\frac{1}{la} + \frac{1}{2x(la)^2} + \frac{1 \cdot 3}{4x^2(la)^3} + \frac{1 \cdot 3 \cdot 5}{8x^3(la)^4} + \text{etc.}\right) : \sqrt{x} + C,$$

ex posteriore autem

$$\int \frac{a^x dx}{\sqrt{x}} = C + \frac{a^x}{\sqrt{x}}\left(\frac{2x}{1} - \frac{4x^2 la}{1 \cdot 3} + \frac{8x^3(la)^2}{1 \cdot 3 \cdot 5} - \frac{16x^4(la)^3}{1 \cdot 3 \cdot 5 \cdot 7} + \text{etc.}\right).$$

SCHOLION 1

228.[1]) Hinc quantitas transcendens $\int \frac{a^x dx}{x}$ per seriem exprimi potest secundum potestates ipsius x progredientem. Cum enim sit

$$a^x = 1 + xla + \frac{x^2(la)^2}{1 \cdot 2} + \frac{x^3(la)^3}{1 \cdot 2 \cdot 3} + \text{etc.},$$

erit

$$\int \frac{a^x dx}{x} = C + lx + \frac{xla}{1} + \frac{x^2(la)^2}{1 \cdot 2 \cdot 2} + \frac{x^3(la)^3}{1 \cdot 2 \cdot 3 \cdot 3} + \frac{x^4(la)^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 4} + \text{etc.},$$

ac si pro a sumamus numerum, cuius logarithmus hyperbolicus est unitas, quem numerum littera e indicemus, habebimus

$$\int \frac{e^x dx}{x} = C + lx + \frac{x}{1} + \frac{1}{2} \cdot \frac{x^2}{1 \cdot 2} + \frac{1}{3} \cdot \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{1}{4} \cdot \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \text{etc.}$$

Atque hinc etiam ponendo $e^x = z$, ut sit $x = lz$, formulam supra memoratam $\frac{dz}{lz}$ per seriem integrare poterimus:

$$\int \frac{dz}{lz} = C + llz + \frac{lz}{1} + \frac{1}{2} \cdot \frac{(lz)^2}{1 \cdot 2} + \frac{1}{3} \cdot \frac{(lz)^3}{1 \cdot 2 \cdot 3} + \frac{1}{4} \cdot \frac{(lz)^4}{1 \cdot 2 \cdot 3 \cdot 4} + \text{etc.};$$

---

1) Cf. MASCHERONII adnotationem I partis prioris; vide notam p. 122. L. S.

quod integrale si debeat evanescere sumto $z = 0$, constans C fit infinita[1]), unde pro reliquis casibus nihil concludi potest. Idem incommodum locum habet, si evanescens reddamus casu $z = 1$, quia $llz = l0$ fit infinitum. Caeterum patet, si integrale sit reale pro valoribus ipsius z unitate minoribus, ubi lz est negativus, tum pro valoribus unitate maioribus fieri imaginarium et vicissim. Hinc ergo natura huius functionis transcendentis parum cognoscitur.

SCHOLION 2

229. Quando vel integratio non succedit vel series ante inventae minus idoneae videntur, hinc quantitatem a^x in seriem resolvendo statim sine aliis subsidiis formulae $a^x X dx$ integrale per seriem exhiberi potest; erit enim

$$\int a^x X dx = \int X dx + \frac{la}{1}\int Xx dx + \frac{(la)^2}{1\cdot 2}\int Xx^2 dx + \frac{(la)^3}{1\cdot 2\cdot 3}\int Xx^3 dx + \text{etc.}$$

Ita, si sit $X = x^n$, habebitur

$$\int a^x x^n dx = C + \frac{x^{n+1}}{n+1} + \frac{x^{n+2} la}{1(n+2)} + \frac{x^{n+3}(la)^2}{1\cdot 2(n+3)} + \frac{x^{n+4}(la)^3}{1\cdot 2\cdot 3(n+4)} + \text{etc.},$$

ubi notandum, si n fuerit numerus integer negativus, puta $n = -i$, loco $\frac{x^{n+i}}{n+i}$ scribi debere lx.

EXEMPLUM 3

230. *Formulae* $\frac{a^x dx}{1-x}$ *integrale per seriem infinitam exprimere.*

Per priorem solutionem obtinemus ob $X = \frac{1}{1-x}$

$$P = \frac{dX}{dx} = \frac{1}{(1-x)^2}, \quad Q = \frac{dP}{dx} = \frac{1\cdot 2}{(1-x)^3}, \quad R = \frac{dQ}{dx} = \frac{1\cdot 2\cdot 3}{(1-x)^4} \quad \text{etc.}$$

hincque sequentem seriem

$$\int \frac{a^x dx}{1-x} = a^x\left(\frac{1}{(1-x)la} - \frac{1}{(1-x)^2(la)^2} + \frac{1\cdot 2}{(1-x)^3(la)^3} - \frac{1\cdot 2\cdot 3}{(1-x)^4(la)^4} + \text{etc.}\right).$$

---

1) Demonstravit L. MASCHERONI in prima *Adnotationum* (vide notam p. 122), si ponatur pro valoribus ipsius z non negativis unitate minoribus

$$\int \frac{dz}{lz} = A + l(-lz) + \frac{lz}{1} + \frac{1}{2}\frac{(lz)^2}{1\cdot 2} + \frac{1}{3}\frac{(lz)^3}{1\cdot 2\cdot 3} + \text{etc.}$$

integrali evanescente sumto $z = 0$, constantem A habere valorem $0{,}577\,215\,664\,901\,532\cdots$, qui valor *Constans* MASCHERONIANA vocari solet. L. S.

Aliae series reperiuntur, si vel a^x vel fractio $\frac{1}{1-x}$ in seriem evolvatur. Commodissima autem videtur, quae seriem fingendo eruitur; brevitatis gratia pro a sumamus numerum e, ut $le = 1$, ac statuatur $dy = \frac{e^x dx}{1-x}$ seu

$$\frac{dy}{dx}(1-x) - 1 - x - \frac{x^2}{1\cdot 2} - \frac{x^3}{1\cdot 2\cdot 3} - \frac{x^4}{1\cdot 2\cdot 3\cdot 4} - \text{etc.} = 0;$$

iam pro y fingatur haec series

$$y = \int \frac{e^x dx}{1-x} = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \text{etc.}$$

eritque facta substitutione

$$\left.\begin{array}{rrrrrl} B + & 2Cx + & 3Dx^2 + & 4Ex^3 + & 5Fx^4 + & \text{etc.} \\ - & B - & 2C - & 3D - & 4E & \\ -1 - & 1 - & \frac{1}{2} - & \frac{1}{6} - & \frac{1}{24} & \end{array}\right\} = 0,$$

unde eliciuntur istae determinationes

$$B = 1, \quad C = \frac{1}{2}(1+1), \quad D = \frac{1}{3}\left(1 + 1 + \frac{1}{2}\right),$$

$$E = \frac{1}{4}\left(1 + 1 + \frac{1}{2} + \frac{1}{6}\right), \quad F = \frac{1}{5}\left(1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24}\right) \quad \text{etc.}$$

PROBLEMA 22

231. *Formulae differentialis $dy = x^{nx} dx$ integrale investigare ac per seriem infinitam exprimere.*

SOLUTIO

Commodius hoc praestari nequit, quam ut formula exponentialis x^{nx} in seriem infinitam convertatur, quae est

$$x^{nx} = 1 + nxlx + \frac{n^2x^2(lx)^2}{1\cdot 2} + \frac{n^3x^3(lx)^3}{1\cdot 2\cdot 3} + \frac{n^4x^4(lx)^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.},$$

qua per dx multiplicata et singulis terminis integratis erit

$$\int dx = x,$$
$$\int x dx lx = x^2\left(\frac{lx}{2} - \frac{1}{2^2}\right),$$
$$\int x^2 dx (lx)^2 = x^3\left(\frac{(lx)^2}{3} - \frac{2lx}{3^2} + \frac{2\cdot 1}{3^3}\right),$$
$$\int x^3 dx (lx)^3 = x^4\left(\frac{(lx)^3}{4} - \frac{3(lx)^2}{4^2} + \frac{3\cdot 2lx}{4^3} - \frac{3\cdot 2\cdot 1}{4^4}\right),$$
$$\int x^4 dx (lx)^4 = x^5\left(\frac{(lx)^4}{5} - \frac{4(lx)^3}{5^2} + \frac{4\cdot 3(lx)^2}{5^3} - \frac{4\cdot 3\cdot 2lx}{5^4} + \frac{4\cdot 3\cdot 2\cdot 1}{5^5}\right)$$
$$\text{etc.}$$

Quare si hae series substituantur et secundum potestates ipsius lx disponantur, integrale quaesitum exprimetur per has innumerabiles series infinitas

$$y = \int x^{nx} dx = x\left(1 - \frac{nx}{2^2} + \frac{n^2x^2}{3^3} - \frac{n^3x^3}{4^4} + \frac{n^4x^4}{5^5} - \text{etc.}\right)$$
$$+ \frac{nx^2 lx}{1}\left(\frac{1}{2^1} - \frac{nx}{3^2} + \frac{n^2x^2}{4^3} - \frac{n^3x^3}{5^4} + \frac{n^4x^4}{6^5} - \text{etc.}\right)$$
$$+ \frac{n^2x^3(lx)^2}{1\cdot 2}\left(\frac{1}{3^1} - \frac{nx}{4^2} + \frac{n^2x^2}{5^3} - \frac{n^3x^3}{6^4} + \frac{n^4x^4}{7^5} - \text{etc.}\right)$$
$$+ \frac{n^3x^4(lx)^3}{1\cdot 2\cdot 3}\left(\frac{1}{4^1} - \frac{nx}{5^2} + \frac{n^2x^2}{6^3} - \frac{n^3x^3}{7^4} + \frac{n^4x^4}{8^5} - \text{etc.}\right)$$
$$\text{etc.},$$

quod integrale ita est sumtum, ut evanescat posito $x = 0$.

COROLLARIUM

232. Hac ergo lege instituta integratione si ponatur $x = 1$, valor integralis $\int x^{nx} dx$ huic seriei aequatur

$$1 - \frac{n}{2^2} + \frac{n^2}{3^3} - \frac{n^3}{4^4} + \frac{n^4}{5^5} - \frac{n^5}{6^6} + \text{etc.},$$

quae ob concinnitatem terminorum omnino est notatu digna.

SCHOLION

233. Eodem modo reperitur integrale huius formulae

$$y=\int x^{nx}x^m dx=\int x^m dx\left(1+nxlx+\frac{n^2x^2(lx)^2}{1\cdot 2}+\frac{n^3x^3(lx)^3}{1\cdot 2\cdot 3}+\text{etc.}\right);$$

erit singulis terminis integrandis

$$\int x^m dx=\frac{x^{m+1}}{m+1},$$

$$\int x^{m+1}dx\,lx \quad = x^{m+2}\left(\frac{lx}{m+2}-\frac{1}{(m+2)^2}\right),$$

$$\int x^{m+2}dx(lx)^2=x^{m+3}\left(\frac{(lx)^2}{m+3}-\frac{2\,lx}{(m+3)^2}+\frac{2\cdot 1}{(m+3)^3}\right),$$

$$\int x^{m+3}dx(lx)^3=x^{m+4}\left(\frac{(lx)^3}{m+4}-\frac{3(lx)^2}{(m+4)^2}+\frac{3\cdot 2lx}{(m+4)^3}-\frac{3\cdot 2\cdot 1}{(m+4)^4}\right)$$

etc.

Quodsi ergo integrale ita determinetur, ut evanescat posito $x=0$, tum vero statuatur $x=1$, pro hoc casu valor formulae integralis $\int x^{nx}x^m dx$ exprimetur hac serie satis memorabili

$$\frac{1}{m+1}-\frac{n}{(m+2)^2}+\frac{nn}{(m+3)^3}-\frac{n^3}{(m+4)^4}+\frac{n^4}{(m+5)^5}-\text{etc.},$$

quae, uti manifestum est, locum habere nequit, quoties m est numerus integer negativus.

Alia exempla formularum exponentialium non adiungo, quia plerumque integralia nimis inconcinne exprimuntur, methodus autem eas tractandi hic sufficienter est exposita. Interim tamen singularem attentionem merentur formulae integrationem absolute admittentes, quae in hac forma continentur $e^x(dP+Pdx)$, cuius integrale manifesto est e^xP. Huiusmodi autem casibus difficile est regulas tradere integrale inveniendi et coniecturae plerumque plurimum est tribuendum. Veluti si proponeretur haec formula

$$\frac{e^x x dx}{(1+x)^2},$$

facile est suspicari integrale, si datur, talem formam esse habiturum

$$\frac{e^x z}{1+x}.$$

Huius ergo differentiale

$$\frac{e^x(dz(1+x)+xzdx)}{(1+x)^2}$$

cum illo comparatum dat

$$dz(1+x)+xzdx=xdx,$$

ubi statim patet esse $z=1$, quod, nisi per se pateret, ex regulis difficulter cognosceretur. Quare transeo ad alterum genus formularum transcendentium iam in Analysin receptarum, quae vel angulos vel sinus tangentesve angulorum complectuntur.

CAPUT V

DE INTEGRATIONE FORMULARUM ANGULOS SINUSVE ANGULORUM IMPLICANTIUM[1])

PROBLEMA 23

234. *Proposita formula differentiali* $X dx \,\text{Ang.}\sin. x$[2]) *eius integrale investigare.*

SOLUTIO

Cum sit
$$d.\,\text{Ang.}\sin. x = \frac{dx}{\sqrt{(1-xx)}},$$
formula proposita ita in factores discerpatur $\text{Ang.}\sin. x \cdot X dx$. Si iam $X dx$ integrationem patiatur sitque $\int X dx = P$, erit nostrum integrale
$$\int X dx \,\text{Ang.}\sin. x = P \,\text{Ang.}\sin. x - \int \frac{P dx}{\sqrt{(1-xx)}}$$
itaque opus reductum est ad integrationem formulae algebraicae, pro qua supra praecepta sunt tradita.

Caeterum si fuerit $X = \frac{1}{\sqrt{(1-xx)}}$, manifestum est integrale fore
$$\int \frac{dx}{\sqrt{(1-xx)}} \,\text{Ang.}\sin. x = \frac{1}{2}(\text{Ang.}\sin. x)^2,$$
quo solo casu quadratum anguli in integrale ingreditur.

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1) Cf. Mascheronii adnotationem II ad cap. V sect. I; vide notam p. 122. L. S.

2) Ang. sin. x, Ang. cos. x etc. idem significant quod Arc. sin. x, Arc. cos. x etc. L. S.

EXEMPLUM 1

235. *Hanc formulam* $dy = x^n dx \,\text{Ang.}\sin . x$ *integrare.*

Cum sit

$$P = \int x^n dx = \frac{x^{n+1}}{n+1},$$

habebimus

$$y = \frac{x^{n+1}}{n+1} \text{Ang.}\sin . x - \frac{1}{n+1} \int \frac{x^{n+1} dx}{\sqrt{(1-xx)}}.$$

Hinc pro variis valoribus ipsius n erunt integralia ope § 120 eruta, ut sequentur

$$\int dx \,\text{Ang.}\sin . x = x \,\text{Ang.}\sin x + \sqrt{(1-xx)} - 1,$$

$$\int x\, dx \,\text{Ang.}\sin . x = \frac{1}{2} xx \,\text{Ang.}\sin . x + \frac{1}{4} x\sqrt{(1-xx)} - \frac{1}{4} \text{Ang.}\sin . x,$$

$$\int x^2 dx \,\text{Ang.}\sin . x = \frac{1}{3} x^3 \text{Ang.}\sin . x + \frac{1}{3}\left(\frac{1}{3}x^2 + \frac{2}{3}\right)\sqrt{(1-xx)} - \frac{1}{3}\cdot\frac{2}{3},$$

$$\int x^3 dx \,\text{Ang.}\sin . x = \frac{1}{4} x^4 \text{Ang.}\sin . x + \frac{1}{4}\left(\frac{1}{4}x^3 + \frac{1\cdot 3}{2\cdot 4}x\right)\sqrt{(1-xx)} - \frac{1}{4}\cdot\frac{1\cdot 3}{2\cdot 4}\text{Ang.}\sin . x,$$

quae ita sunt sumta, ut evanescant posito $x = 0$.

EXEMPLUM 2

236. *Hanc formulam* $dy = \frac{x dx}{\sqrt{(1-xx)}} \text{Ang.}\sin . x$ *integrare.*

Cum sit

$$\int \frac{x dx}{\sqrt{(1-xx)}} = -\sqrt{(1-xx)} = P,$$

erit integrale quaesitum

$$y = C - \sqrt{(1-xx)}\,\text{Ang.}\sin . x + \int \frac{dx\sqrt{(1-xx)}}{\sqrt{(1-xx)}}$$

sicque habebitur

$$y = \int \frac{x dx}{\sqrt{(1-xx)}} \text{Ang.}\sin . x = C - \sqrt{(1-xx)}\,\text{Ang.}\sin . x + x.$$

EXEMPLUM 3

237. *Hanc formulam* $dy = \frac{dx}{(1-xx)^{\frac{3}{2}}} \text{Ang.}\sin. x$ *integrare.*

Hic est

$$P = \int \frac{dx}{(1-xx)^{\frac{3}{2}}} = \frac{x}{\sqrt{(1-xx)}},$$

unde fit

$$y = \frac{x}{\sqrt{(1-xx)}} \text{Ang.}\sin. x - \int \frac{x dx}{1-xx}$$

seu

$$y = \int \frac{dx}{(1-xx)^{\frac{3}{2}}} \text{Ang.}\sin. x = \frac{x}{\sqrt{(1-xx)}} \text{Ang.}\sin. x + l\sqrt{(1-xx)},$$

quod integrale evanescit posito $x = 0$.

SCHOLION

238. Simili modo integratur formula $dy = X dx \,\text{Ang.}\cos. x$. Cum enim sit

$$d.\text{Ang.}\cos. x = \frac{-dx}{\sqrt{(1-xx)}},$$

si ponamus $\int X dx = P$, erit

$$y = P \,\text{Ang.}\cos. x + \int \frac{P dx}{\sqrt{(1-xx)}}.$$

Quin etiam si proponatur formula $dy = X dx \,\text{Ang.}\,\text{tang.}\, x$, quia est

$$d.\text{Ang.}\,\text{tang.}\, x = \frac{dx}{1+xx},$$

posito $\int X dx = P$ erit hoc integrale

$$y = \int X dx \,\text{Ang.}\,\text{tang.}\, x = P \,\text{Ang.}\,\text{tang.}\, x - \int \frac{P dx}{1+xx}.$$

Quoties ergo $\int X dx$ algebraice dari potest, toties integratio reducitur ad formulam algebraicam sicque negotium confectum est habendum. Cum igitur in his formulis angulus, cuius sinus, cosinus vel tangens erat $= x$, inesset, consideremus etiam eiusmodi formulas, in quas quadratum huius anguli altiorve potestas ingreditur.

PROBLEMA 24

239. *Denotet φ angulum, cuius sinus tangensve est functio quaedam ipsius x, unde fiat $d\varphi = udx$, propositaque sit haec formula $dy = Xdx \cdot \varphi^n$, quam integrare oporteat.*

SOLUTIO

Sit $\int Xdx = P$, ut habeamus $dy = \varphi^n dP$, eritque integrando

$$y = \varphi^n P - n\int \varphi^{n-1} Pudx.$$

Iam simili modo sit $\int Pudx = Q$; erit

$$\int \varphi^{n-1} Pudx = \varphi^{n-1} Q - (n-1)\int \varphi^{n-2} Qudx;$$

tum posito $\int Qudx = R$ erit

$$\int \varphi^{n-2} Qudx = \varphi^{n-2} R - (n-2)\int \varphi^{n-3} Rudx.$$

Hocque modo potestas anguli φ continuo deprimitur, donec tandem ad formulam ab angulo φ liberam perveniatur; id quod semper eveniet, dummodo n sit numerus integer positivus et haec integralia continuo sumere liceat

$$\int Xdx = P, \quad \int Pudx = Q, \quad \int Qudx = R \quad \text{etc.},$$

quae integrationes si non succedant, frustra integratio suscipitur.

EXEMPLUM

240. *Sit φ angulus, cuius sinus $= x$, ut sit $d\varphi = \frac{dx}{\sqrt{(1-xx)}}$; integrare formulam $dy = \varphi^n dx$.*

Erit ergo

$$X = 1, \quad P = x, \quad Q = \int \frac{Pdx}{\sqrt{(1-xx)}} = -\sqrt{(1-xx)}, \quad R = \int \frac{Qdx}{\sqrt{(1-xx)}} = -x,$$

$$S = \int \frac{Rdx}{\sqrt{(1-xx)}} = \sqrt{(1-xx)}, \quad T = x \quad \text{etc.},$$

quibus valoribus inventis reperietur

$$y = \int \varphi^n dx = \varphi^n x + n\varphi^{n-1}\sqrt{(1-xx)} - n(n-1)\varphi^{n-2}x$$
$$- n(n-1)(n-2)\varphi^{n-3}\sqrt{(1-xx)} + \text{etc.}$$

Pro variis ergo valoribus exponentis n habebimus

$$\int \varphi\, dx = \varphi\, x + \sqrt{(1-xx)} - 1,$$
$$\int \varphi^2 dx = \varphi^2 x + 2\varphi\sqrt{(1-xx)} - 2\cdot 1x,$$
$$\int \varphi^3 dx = \varphi^3 x + 3\varphi^2\sqrt{(1-xx)} - 3\cdot 2\varphi x - 3\cdot 2\cdot 1\sqrt{(1-xx)} + 6$$
$$\text{etc.}$$

integralibus ita determinatis, ut evanescant posito $x = 0$.

SCHOLION

241. Si sit $Xdx = udx = d\varphi$, formulae $\varphi^n d\varphi$ integrale est $\frac{1}{n+1}\varphi^{n+1}$; similique modo si fuerit Φ functio quaecunque anguli φ, formulae $\Phi u dx = \Phi d\varphi$ integratio nihil habet difficultatis. Multo latius patent formulae sinus cosinusve angulorum et tangentes implicantes, quarum integratio per universam Analysin amplissimum habet usum, cum praecipue Theoria Astronomiae ad huiusmodi formulas sit reducta. Prima autem fundamenta peti debent ex calculo differentiali; unde cum sit

$$d.\sin. n\varphi = nd\varphi\cos. n\varphi,\quad d.\cos. n\varphi = -nd\varphi\sin. n\varphi,\quad d.\text{tang.} n\varphi = \frac{nd\varphi}{\cos. n\varphi^2},$$

$$d.\cot. n\varphi = \frac{-nd\varphi}{\sin. n\varphi^2},\quad d.\frac{1}{\sin. n\varphi} = \frac{-nd\varphi\cos. n\varphi}{\sin. n\varphi^2},\quad d.\frac{1}{\cos. n\varphi} = \frac{nd\varphi\sin. n\varphi}{\cos. n\varphi^2},$$

nanciscimur has integrationes elementares

$$\int d\varphi\cos. n\varphi = \frac{1}{n}\sin. n\varphi,\quad \int d\varphi\sin. n\varphi = -\frac{1}{n}\cos. n\varphi,$$

$$\int \frac{d\varphi}{\cos. n\varphi^2} = \frac{1}{n}\text{tang.} n\varphi,\quad \int \frac{d\varphi}{\sin. n\varphi^2} = -\frac{1}{n}\cot. n\varphi,$$

$$\int \frac{d\varphi\cos. n\varphi}{\sin. n\varphi^2} = -\frac{1}{n\sin. n\varphi},\quad \int \frac{d\varphi\sin. n\varphi}{\cos. n\varphi^2} = \frac{1}{n\cos. n\varphi},$$

unde statim huiusmodi formularum differentialium

$$d\varphi(A + B\cos.\varphi + C\cos.2\varphi + D\cos.3\varphi + E\cos.4\varphi + \text{etc.})$$

[integratio] consequitur, cum integrale manifesto sit

$$A\varphi + B\sin.\varphi + \frac{1}{2}C\sin.2\varphi + \frac{1}{3}D\sin.3\varphi + \frac{1}{4}E\sin.4\varphi + \text{etc.}$$

Deinde etiam in subsidium vocari convenit, quae in elementis de angulorum compositione traduntur, scilicet

$$\sin.\alpha\cdot\sin.\beta = \frac{1}{2}\cos.(\alpha-\beta) - \frac{1}{2}\cos.(\alpha+\beta),$$

$$\cos.\alpha\cdot\cos.\beta = \frac{1}{2}\cos.(\alpha-\beta) + \frac{1}{2}\cos.(\alpha+\beta),$$

$$\sin.\alpha\cdot\cos.\beta = \frac{1}{2}\sin.(\alpha+\beta) + \frac{1}{2}\sin.(\alpha-\beta)$$

$$= \frac{1}{2}\sin.(\alpha+\beta) - \frac{1}{2}\sin.(\beta-\alpha),$$

unde producta plurium sinuum et cosinuum in simplices sinus cosinusve resolvuntur.

PROBLEMA 25

242. *Formulae* $d\varphi\sin.\varphi^n$ *integrale investigare.*

SOLUTIO

Repraesentetur in hos factores resoluta $\sin.\varphi^{n-1}\cdot d\varphi\sin.\varphi$, et quia

$$\int d\varphi\sin.\varphi = -\cos.\varphi,$$

erit

$$\int d\varphi\sin.\varphi^n = -\sin.\varphi^{n-1}\cos.\varphi + (n-1)\int d\varphi\sin.\varphi^{n-2}\cos.\varphi^2.$$

Hinc ob $\cos.\varphi^2 = 1 - \sin.\varphi^2$ habebitur

$$\int d\varphi\sin.\varphi^n = -\sin.\varphi^{n-1}\cos.\varphi + (n-1)\int d\varphi\sin.\varphi^{n-2} - (n-1)\int d\varphi\sin.\varphi^n,$$

ubi cum postrema formula ipsi propositae sit similis, hinc colligitur ista reductio

$$\int d\varphi \sin.\varphi^n = -\frac{1}{n}\sin.\varphi^{n-1}\cos.\varphi + \frac{n-1}{n}\int d\varphi \sin.\varphi^{n-2},$$

qua integratio ad hanc formulam simpliciorem $d\varphi \sin.\varphi^{n-2}$ revocatur. Cum igitur casus simplicissimi constent

$$\int d\varphi \sin.\varphi^0 = \varphi \quad \text{et} \quad \int d\varphi \sin.\varphi = -\cos.\varphi,$$

hinc via ad continuo maiores exponentes n paratur:

$$\int d\varphi \sin.\varphi^0 = \varphi,$$

$$\int d\varphi \sin.\varphi = -\cos.\varphi,$$

$$\int d\varphi \sin.\varphi^2 = -\frac{1}{2}\sin.\varphi\cos.\varphi + \frac{1}{2}\varphi,$$

$$\int d\varphi \sin.\varphi^3 = -\frac{1}{3}\sin.\varphi^2\cos.\varphi - \frac{2}{3}\cos.\varphi,$$

$$\int d\varphi \sin.\varphi^4 = -\frac{1}{4}\sin.\varphi^3\cos.\varphi - \frac{1\cdot 3}{2\cdot 4}\sin.\varphi\cos.\varphi + \frac{1\cdot 3}{2\cdot 4}\varphi,$$

$$\int d\varphi \sin.\varphi^5 = -\frac{1}{5}\sin.\varphi^4\cos.\varphi - \frac{1\cdot 4}{3\cdot 5}\sin.\varphi^2\cos.\varphi - \frac{2\cdot 4}{3\cdot 5}\cos.\varphi,$$

$$\int d\varphi \sin.\varphi^6 = -\frac{1}{6}\sin.\varphi^5\cos.\varphi - \frac{1\cdot 5}{4\cdot 6}\sin.\varphi^3\cos.\varphi - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\sin.\varphi\cos.\varphi + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\varphi$$

etc.

COROLLARIUM 1

243. Quoties n est numerus impar, integrale per solum sinum et cosinum exhibetur, at si n est numerus par, integrale insuper ipsum angulum involvit ideoque est functio transcendens.

COROLLARIUM 2

244. Casibus ergo, quibus n est numerus impar, id imprimis notari convenit, etiamsi angulus seu arcus φ in infinitum crescat, integrale tamen nunquam ultra certum limitem excrescere posse, cum tamen, si n sit numerus par, etiam in infinitum excrescat.

SCHOLION

245. Simili modo formula $d\varphi\cos.\varphi^n$ tractatur, quae in hos factores resoluta $\cos.\varphi^{n-1}\cdot d\varphi\cos.\varphi$ praebet

$$\int d\varphi\cos.\varphi^n = \cos.\varphi^{n-1}\sin.\varphi + (n-1)\int d\varphi\cos.\varphi^{n-2}\sin.\varphi^2$$
$$= \cos.\varphi^{n-1}\sin.\varphi + (n-1)\int d\varphi\cos.\varphi^{n-2} - (n-1)\int d\varphi\cos.\varphi^n,$$

unde, cum postrema formula propositae sit similis, colligitur

$$\int d\varphi\cos.\varphi^n = \frac{1}{n}\sin.\varphi\cos.\varphi^{n-1} + \frac{n-1}{n}\int d\varphi\cos.\varphi^{n-2}.$$

Quare cum casibus $n=0$ et $n=1$ integratio sit in promtu, ad altiores potestates patet progressio:

$$\int d\varphi\cos.\varphi^0 = \varphi,$$
$$\int d\varphi\cos.\varphi = \sin.\varphi,$$
$$\int d\varphi\cos.\varphi^2 = \frac{1}{2}\sin.\varphi\cos.\varphi + \frac{1}{2}\varphi,$$
$$\int d\varphi\cos.\varphi^3 = \frac{1}{3}\sin.\varphi\cos.\varphi^2 + \frac{2}{3}\sin.\varphi,$$
$$\int d\varphi\cos.\varphi^4 = \frac{1}{4}\sin.\varphi\cos.\varphi^3 + \frac{1\cdot 3}{2\cdot 4}\sin.\varphi\cos.\varphi + \frac{1\cdot 3}{2\cdot 4}\varphi,$$
$$\int d\varphi\cos.\varphi^5 = \frac{1}{5}\sin.\varphi\cos.\varphi^4 + \frac{1\cdot 4}{3\cdot 5}\sin.\varphi\cos.\varphi^2 + \frac{2\cdot 4}{3\cdot 5}\sin.\varphi,$$
$$\int d\varphi\cos.\varphi^6 = \frac{1}{6}\sin.\varphi\cos.\varphi^5 + \frac{1\cdot 5}{4\cdot 6}\sin.\varphi\cos.\varphi^3 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\sin.\varphi\cos.\varphi + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\varphi$$

etc.

PROBLEMA 26

246. *Formulae* $d\varphi\sin.\varphi^m\cos.\varphi^n$ *integrale invenire.*

SOLUTIO

Quo hoc facilius praestetur, consideremus factum $\sin.\varphi^\mu\cos.\varphi^\nu$, quod differentiatum fit $\mu d\varphi\sin.\varphi^{\mu-1}\cos.\varphi^{\nu+1} - \nu d\varphi\sin.\varphi^{\mu+1}\cos.\varphi^{\nu-1}$. Iam prout vel in parte priori $\cos.\varphi^2 = 1-\sin.\varphi^2$ vel in posteriori $\sin.\varphi^2 = 1-\cos.\varphi_2$

statuitur, oritur vel

$$d.\sin.\varphi^\mu\cos.\varphi^\nu = +\mu d\varphi\sin.\varphi^{\mu-1}\cos.\varphi^{\nu-1} - (\mu+\nu)d\varphi\sin.\varphi^{\mu+1}\cos.\varphi^{\nu-1}$$

vel

$$d.\sin.\varphi^\mu\cos.\varphi^\nu = -\nu d\varphi\sin.\varphi^{\mu-1}\cos.\varphi^{\nu-1} + (\mu+\nu)d\varphi\sin.\varphi^{\mu-1}\cos.\varphi^{\nu+1}.$$

Hinc igitur duplicem reductionem adipiscimur

$$\text{I.}\quad \int d\varphi\sin.\varphi^{\mu+1}\cos.\varphi^{\nu-1} = -\frac{1}{\mu+\nu}\sin.\varphi^\mu\cos.\varphi^\nu + \frac{\mu}{\mu+\nu}\int d\varphi\sin.\varphi^{\mu-1}\cos.\varphi^{\nu-1},$$

$$\text{II.}\quad \int d\varphi\sin.\varphi^{\mu-1}\cos.\varphi^{\nu+1} = \frac{1}{\mu+\nu}\sin.\varphi^\mu\cos.\varphi^\nu + \frac{\nu}{\mu+\nu}\int d\varphi\sin.\varphi^{\mu-1}\cos.\varphi^{\nu-1}.$$

Quare formula proposita $\int d\varphi\sin.\varphi^m\cos.\varphi^n$ successive continuo ad simpliciores potestates tam ipsius $\sin.\varphi$ quam ipsius $\cos.\varphi$ reducitur, donec alter vel penitus abeat vel simpliciter adsit, quo casu integratio per se patet, cum sit

$$\int d\varphi\sin.\varphi^m\cos.\varphi = +\frac{1}{m+1}\sin.\varphi^{m+1} \quad\text{et}\quad \int d\varphi\sin.\varphi\cos.\varphi^n = -\frac{1}{n+1}\cos.\varphi^{n+1}.$$

EXEMPLUM

247. *Formulae* $d\varphi\sin.\varphi^8\cos.\varphi^7$ *integrale invenire.*

Per priorem reductionem ob $\mu = 7$ et $\nu = 8$ impetramus

$$\int d\varphi\sin.\varphi^8\cos.\varphi^7 = -\frac{1}{15}\sin.\varphi^7\cos.\varphi^8 + \frac{7}{15}\int d\varphi\sin.\varphi^6\cos.\varphi^7;$$

istam per posteriorem reductionem tractemus

$$\int d\varphi\sin.\varphi^6\cos.\varphi^7 = \frac{1}{13}\sin.\varphi^7\cos.\varphi^6 + \frac{6}{13}\int d\varphi\sin.\varphi^6\cos.\varphi^5;$$

hoc modo ulterius progrediamur

$$\int d\varphi\sin.\varphi^6\cos.\varphi^5 = -\frac{1}{11}\sin.\varphi^5\cos.\varphi^6 + \frac{5}{11}\int d\varphi\sin.\varphi^4\cos.\varphi^5,$$

$$\int d\varphi\sin.\varphi^4\cos.\varphi^5 = \frac{1}{9}\sin.\varphi^5\cos.\varphi^4 + \frac{4}{9}\int d\varphi\sin.\varphi^4\cos.\varphi^3,$$

$$\int d\varphi\sin.\varphi^4\cos.\varphi^3 = -\frac{1}{7}\sin.\varphi^3\cos.\varphi^4 + \frac{3}{7}\int d\varphi\sin.\varphi^2\cos.\varphi^3,$$

$$\int d\varphi\sin.\varphi^2\cos.\varphi^3 = \frac{1}{5}\sin.\varphi^3\cos.\varphi^2 + \frac{2}{5}\int d\varphi\sin.\varphi^2\cos.\varphi,$$

$$\int d\varphi\sin.\varphi^2\cos.\varphi = -\frac{1}{3}\sin.\varphi\cos.\varphi^2 + \frac{1}{3}\int d\varphi\cos.\varphi \qquad \left(+\frac{1}{3}\sin\varphi\right).$$

Ex his colligitur formulae propositae integrale

$$\int d\varphi \sin.\varphi^8 \cos.\varphi^7 = -\frac{1}{15}\sin.\varphi^7\cos.\varphi^8 + \frac{1\cdot 7}{15\cdot 13}\sin.\varphi^7\cos.\varphi^6 - \frac{1\cdot 7\cdot 6}{15\cdot 13\cdot 11}\sin.\varphi^5\cos.\varphi^6$$
$$+\frac{1\cdot 7\cdot 6\cdot 5}{15\cdot 13\cdot 11\cdot 9}\sin.\varphi^5\cos.\varphi^4 - \frac{1\cdot 7\cdot 6\cdot 5\cdot 4}{15\cdot 13\cdot 11\cdot 9\cdot 7}\sin.\varphi^3\cos.\varphi^4$$
$$+\frac{1\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3}{15\cdot 13\cdot 11\cdot 9\cdot 7\cdot 5}\sin.\varphi^3\cos.\varphi^2 - \frac{1\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}{15\cdot 13\cdot 11\cdot 9\cdot 7\cdot 5\cdot 3}\sin.\varphi\cos.\varphi^2$$
$$+\frac{1\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}{15\cdot 13\cdot 11\cdot 9\cdot 7\cdot 5\cdot 3}\sin.\varphi.$$

SCHOLION

248. Quando autem huiusmodi casus occurrunt, semper praestat productum $\sin.\varphi^m\cos.\varphi^n$ in sinus vel cosinus angulorum multiplorum resolvere, quo facto singulae partes facillime integrantur. Caeterum hic brevitatis gratia angulum simpliciter littera φ indicavi nihiloque res foret generalior, si per $\alpha\varphi+\beta$ exprimeretur, quemadmodum etiam ante haec expressio Ang. sin. x aeque late patet, ac si loco x functio quaecunque scriberetur. Contemplemur ergo eiusmodi formulas, in quibus sinus cosinusve denominatorem occupant, ubi quidem simplicissimae sunt

$$\text{I. } \frac{d\varphi}{\sin.\varphi},\quad \text{II. } \frac{d\varphi}{\cos.\varphi},\quad \text{III. } \frac{d\varphi\cos.\varphi}{\sin.\varphi},\quad \text{IV. } \frac{d\varphi\sin.\varphi}{\cos.\varphi},$$

quarum integralia imprimis nosse oportet. Pro prima adhibeantur hae transformationes

$$\frac{d\varphi}{\sin.\varphi} = \frac{d\varphi\sin.\varphi}{\sin.\varphi^2} = \frac{d\varphi\sin.\varphi}{1-\cos.\varphi^2} = \frac{-dx}{1-xx}\quad (\text{posito } \cos.\varphi = x),$$

unde fit

$$\int\frac{d\varphi}{\sin.\varphi} = -\frac{1}{2}l\frac{1+x}{1-x} = -\frac{1}{2}l\frac{1+\cos.\varphi}{1-\cos.\varphi}.$$

Pro secunda

$$\frac{d\varphi}{\cos.\varphi} = \frac{d\varphi\cos.\varphi}{\cos.\varphi^2} = \frac{d\varphi\cos.\varphi}{1-\sin\varphi^2} = \frac{dx}{1-xx}\quad (\text{posito } \sin.\varphi = x),$$

ergo

$$\int\frac{d\varphi}{\cos.\varphi} = \frac{1}{2}l\frac{1+x}{1-x} = \frac{1}{2}l\frac{1+\sin.\varphi}{1-\sin.\varphi}.$$

Tertiae et quartae integratio manifesto logarithmis conficitur; quare haec integralia probe notasse iuvabit

$$\text{I.}\quad \int\frac{d\varphi}{\sin.\varphi} = -\frac{1}{2}l\frac{1+\cos.\varphi}{1-\cos.\varphi} = l\frac{\sqrt{(1-\cos.\varphi)}}{\sqrt{(1+\cos.\varphi)}} = l\,\text{tang.}\,\frac{1}{2}\varphi,$$

$$\text{II.}\quad \int\frac{d\varphi}{\cos.\varphi} = \frac{1}{2}l\frac{1+\sin.\varphi}{1-\sin.\varphi} = l\frac{\sqrt{(1+\sin.\varphi)}}{\sqrt{(1-\sin.\varphi)}} = l\,\text{tang.}\left(45^0+\frac{1}{2}\varphi\right),$$

$$\text{III.}\quad \int\frac{d\varphi\cos.\varphi}{\sin.\varphi} = l\sin.\varphi = \int\frac{d\varphi}{\text{tang.}\,\varphi} = \int d\varphi\cot.\varphi,$$

$$\text{IV.}\quad \int\frac{d\varphi\sin.\varphi}{\cos.\varphi} = -l\cos.\varphi = \int d\varphi\,\text{tang.}\,\varphi;$$

hincque sequitur III + IV

$$\int\frac{d\varphi}{\sin.\varphi\cos.\varphi} = l\frac{\sin.\varphi}{\cos.\varphi} = l\,\text{tang.}\,\varphi.$$

PROBLEMA 27

249. *Formularum* $\frac{d\varphi\sin.\varphi^m}{\cos.\varphi^n}$ *et* $\frac{d\varphi\cos.\varphi^m}{\sin.\varphi^n}$ *integralia investigare.*

SOLUTIO

Primo statim perspicitur alteram formulam in alteram transmutari posito $\varphi = 90^0 - \psi$, quia tum fit $\sin.\varphi = \cos.\psi$ et $\cos.\varphi = \sin.\psi$, dummodo notetur fore $d\varphi = -d\psi$. Quare sufficit priorem tantum tractasse. Reductio autem prior § 246 data sumto $\mu + 1 = m$ et $\nu - 1 = -n$ praebet

$$\int\frac{d\varphi\sin.\varphi^m}{\cos.\varphi^n} = -\frac{1}{m-n}\cdot\frac{\sin.\varphi^{m-1}}{\cos.\varphi^{n-1}} + \frac{m-1}{m-n}\int\frac{d\varphi\sin.\varphi^{m-2}}{\cos.\varphi^n},$$

quo pacto in numeratore exponens ipsius $\sin.\varphi$ continuo binario deprimitur, ita ut tandem perveniatur vel ad $\int\frac{d\varphi}{\cos.\varphi^n}$ vel ad $\int\frac{d\varphi\sin.\varphi}{\cos.\varphi^n} = \frac{1}{(n-1)\cos.\varphi^{n-1}}$ ideoque sola formula $\int\frac{d\varphi}{\cos.\varphi^n}$ tractanda supersit. Altera autem reductio ibidem tradita (§ 246) sumto $\mu - 1 = m$ et $\nu - 1 = -n$ dat

$$\int\frac{d\varphi\sin.\varphi^m}{\cos.\varphi^{n-2}}=\frac{1}{m-n+2}\cdot\frac{\sin.\varphi^{m+1}}{\cos.\varphi^{n-1}}-\frac{n-1}{m-n+2}\int\frac{d\varphi\sin.\varphi^m}{\cos.\varphi^n},$$

unde colligitur

$$\int\frac{d\varphi\sin.\varphi^m}{\cos.\varphi^n}=\frac{1}{n-1}\cdot\frac{\sin.\varphi^{m+1}}{\cos.\varphi^{n-1}}-\frac{m-n+2}{n-1}\int\frac{d\varphi\sin.\varphi^m}{\cos.\varphi^{n-2}},$$

cuius reductionis ope exponens ipsius $\cos.\varphi$ in denominatore continuo binario deprimitur, ita ut tandem vel ad $\int d\varphi\sin.\varphi^m$ vel ad $\int\frac{d\varphi\sin.\varphi^m}{\cos.\varphi}$ perveniatur. Illius integratio iam supra est monstrata, huius vero forma, si $m>1$, per priorem reductionem tandem vel ad $\int\frac{d\varphi}{\cos.\varphi}$ vel ad $\int\frac{d\varphi\sin.\varphi}{\cos.\varphi}$ revocatur; illius autem integrale est $l\,\mathrm{tang.}\left(45^0+\frac{1}{2}\varphi\right)$, huius vero $-l\cos.\varphi$.

COROLLARIUM 1

250. Prior reductio non habet locum, quoties est $m=n$, hoc scilicet casu formula $\int\frac{d\varphi\sin.\varphi^n}{\cos.\varphi^n}$ non reduci potest ad formulam $\int\frac{d\varphi\sin.\varphi^{n-2}}{\cos.\varphi^n}$. Altera autem reductione semper uti licet; etsi enim casus $n=1$ inde excluditur, eius tamen integratio per priorem effici potest.

COROLLARIUM 2

251. Ratio autem illius exclusionis in hoc est posita, quod formula $\int\frac{d\varphi\sin.\varphi^{n-2}}{\cos.\varphi^n}$ est absolute integrabilis habens integrale $=\frac{1}{n-1}\cdot\frac{\sin.\varphi^{n-1}}{\cos.\varphi^{n-1}}$. Erit ergo pro his casibus

$$\int\frac{d\varphi}{\cos.\varphi^2}=\frac{\sin.\varphi}{\cos.\varphi}=\mathrm{tang.}\,\varphi,\quad\int\frac{d\varphi\sin.\varphi}{\cos.\varphi^3}=\frac{1}{2}\cdot\frac{\sin.\varphi^2}{\cos.\varphi^2}=\frac{1}{2}\mathrm{tang.}\,\varphi^2,$$

$$\int\frac{d\varphi\sin.\varphi^2}{\cos.\varphi^4}=\frac{1}{3}\cdot\frac{\sin.\varphi^3}{\cos.\varphi^3}=\frac{1}{3}\mathrm{tang.}\,\varphi^3,\quad\int\frac{d\varphi\sin.\varphi^3}{\cos.\varphi^5}=\frac{1}{4}\cdot\frac{\sin.\varphi^4}{\cos.\varphi^4}=\frac{1}{4}\mathrm{tang.}\,\varphi^4.$$

EXEMPLUM 1

252. *Formulae* $\frac{d\varphi\sin.\varphi^m}{\cos.\varphi}$ *integrale assignare.*

Prior reductio dat

$$\int\frac{d\varphi\sin.\varphi^m}{\cos.\varphi}=\frac{-1}{m-1}\sin.\varphi^{m-1}+\int\frac{d\varphi\sin.\varphi^{m-2}}{\cos.\varphi}.$$

Hinc a casibus per se notis incipiendo habebimus

$$\int \frac{d\varphi}{\cos.\varphi} = l\,\mathrm{tang.}\left(45^0 + \frac{1}{2}\varphi\right),$$

$$\int \frac{d\varphi \sin.\varphi}{\cos.\varphi} = -\,l\cos.\varphi = l\sec.\varphi,$$

$$\int \frac{d\varphi \sin.\varphi^2}{\cos.\varphi} = -\sin.\varphi + \int \frac{d\varphi}{\cos.\varphi},$$

$$\int \frac{d\varphi \sin.\varphi^3}{\cos.\varphi} = -\frac{1}{2}\sin.\varphi^2 + l\sec.\varphi,$$

$$\int \frac{d\varphi \sin.\varphi^4}{\cos.\varphi} = -\frac{1}{3}\sin.\varphi^3 - \sin.\varphi + \int \frac{d\varphi}{\cos.\varphi},$$

$$\int \frac{d\varphi \sin.\varphi^5}{\cos.\varphi} = -\frac{1}{4}\sin.\varphi^4 - \frac{1}{2}\sin.\varphi^2 + l\sec.\varphi,$$

$$\int \frac{d\varphi \sin.\varphi^6}{\cos.\varphi} = -\frac{1}{5}\sin.\varphi^5 - \frac{1}{3}\sin.\varphi^3 - \sin.\varphi + \int \frac{d\varphi}{\cos.\varphi},$$

$$\int \frac{d\varphi \sin.\varphi^7}{\cos.\varphi} = -\frac{1}{6}\sin.\varphi^6 - \frac{1}{4}\sin.\varphi^4 - \frac{1}{2}\sin.\varphi^2 + l\sec.\varphi$$

etc.

SCHOLION

253. Pro reliquis casibus denominatoris totum negotium conficietur his reductionibus

$$\int \frac{d\varphi \sin.\varphi^m}{\cos.\varphi^2} = \frac{\sin.\varphi^{m+1}}{\cos.\varphi} - m\int d\varphi \sin.\varphi^m,$$

$$\int \frac{d\varphi \sin.\varphi^m}{\cos.\varphi^3} = \frac{1}{2}\cdot\frac{\sin.\varphi^{m+1}}{\cos.\varphi^2} - \frac{m-1}{2}\int \frac{d\varphi \sin.\varphi^m}{\cos.\varphi},$$

$$\int \frac{d\varphi \sin.\varphi^m}{\cos.\varphi^4} = \frac{1}{3}\cdot\frac{\sin.\varphi^{m+1}}{\cos.\varphi^3} - \frac{m-2}{3}\int \frac{d\varphi \sin.\varphi^m}{\cos.\varphi^2},$$

$$\int \frac{d\varphi \sin.\varphi^m}{\cos.\varphi^5} = \frac{1}{4}\cdot\frac{\sin.\varphi^{m+1}}{\cos.\varphi^4} - \frac{m-3}{4}\int \frac{d\varphi \sin.\varphi^m}{\cos.\varphi^3}$$

etc.

EXEMPLUM 2

254. *Formulae* $\frac{d\varphi}{\cos.\varphi^n}$ *integrale assignare.*

Altera reductio ob $m=0$ fit

$$\int\frac{d\varphi}{\cos.\varphi^n}=\frac{1}{n-1}\cdot\frac{\sin.\varphi}{\cos.\varphi^{n-1}}+\frac{n-2}{n-1}\int\frac{d\varphi}{\cos.\varphi^{n-2}};$$

quia iam casus simplicissimi

$$\int d\varphi=\varphi \quad \text{et} \quad \int\frac{d\varphi}{\cos.\varphi}=l\,\text{tang.}\left(45^0+\frac{1}{2}\varphi\right)$$

sunt cogniti, ad eos sequentes omnes revocabuntur

$$\int\frac{d\varphi}{\cos.\varphi^2}=\frac{\sin.\varphi}{\cos.\varphi},$$

$$\int\frac{d\varphi}{\cos.\varphi^3}=\frac{1}{2}\cdot\frac{\sin.\varphi}{\cos.\varphi^2}+\frac{1}{2}\int\frac{d\varphi}{\cos.\varphi},$$

$$\int\frac{d\varphi}{\cos.\varphi^4}=\frac{1}{3}\cdot\frac{\sin.\varphi}{\cos.\varphi^3}+\frac{2}{3}\cdot\frac{\sin.\varphi}{\cos.\varphi},$$

$$\int\frac{d\varphi}{\cos.\varphi^5}=\frac{1}{4}\cdot\frac{\sin.\varphi}{\cos.\varphi^4}+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{\sin.\varphi}{\cos.\varphi^2}+\frac{1\cdot 3}{2\cdot 4}\int\frac{d\varphi}{\cos.\varphi},$$

$$\int\frac{d\varphi}{\cos.\varphi^6}=\frac{1}{5}\cdot\frac{\sin.\varphi}{\cos.\varphi^5}+\frac{1\cdot 4}{3\cdot 5}\cdot\frac{\sin.\varphi}{\cos.\varphi^3}+\frac{2\cdot 4}{3\cdot 5}\cdot\frac{\sin.\varphi}{\cos.\varphi}$$

etc.

COROLLARIUM 1

255. Simili modo habebimus has integrationes

$$\int\frac{d\varphi}{\sin.\varphi}=l\,\text{tang.}\frac{1}{2}\varphi,\quad \int\frac{d\varphi}{\sin.\varphi^2}=-\frac{\cos.\varphi}{\sin.\varphi},$$

$$\int\frac{d\varphi}{\sin.\varphi^3}=-\frac{1}{2}\cdot\frac{\cos.\varphi}{\sin.\varphi^2}+\frac{1}{2}\int\frac{d\varphi}{\sin.\varphi},$$

$$\int\frac{d\varphi}{\sin.\varphi^4}=-\frac{1}{3}\cdot\frac{\cos.\varphi}{\sin.\varphi^3}-\frac{2}{3}\cdot\frac{\cos.\varphi}{\sin.\varphi},$$

$$\int\frac{d\varphi}{\sin.\varphi^5}=-\frac{1}{4}\cdot\frac{\cos.\varphi}{\sin.\varphi^4}-\frac{1\cdot 3}{2\cdot 4}\cdot\frac{\cos.\varphi}{\sin.\varphi^2}+\frac{1\cdot 3}{2\cdot 4}\int\frac{d\varphi}{\sin.\varphi}$$

etc.

COROLLARIUM 2

256. Deinde est

$$\int\frac{d\varphi\sin.\varphi}{\cos.\varphi^n}=\frac{1}{n-1}\cdot\frac{1}{\cos.\varphi^{n-1}}\quad\text{et}\quad\int\frac{d\varphi\cos.\varphi}{\sin.\varphi^n}=\frac{-1}{n-1}\cdot\frac{1}{\sin.\varphi^{n-1}}.$$

Porro

$$\int\frac{d\varphi\sin.\varphi^2}{\cos.\varphi^n}=\int\frac{d\varphi}{\cos.\varphi^n}-\int\frac{d\varphi}{\cos.\varphi^{n-2}},\qquad\int\frac{d\varphi\cos.\varphi^2}{\sin.\varphi^n}=\int\frac{d\varphi}{\sin.\varphi^n}-\int\frac{d\varphi}{\sin.\varphi^{n-2}}$$

et

$$\int\frac{d\varphi\sin.\varphi^3}{\cos.\varphi^n}=\int\frac{d\varphi\sin.\varphi}{\cos.\varphi^n}-\int\frac{d\varphi\sin.\varphi}{\cos.\varphi^{n-2}},\quad\int\frac{d\varphi\cos.\varphi^3}{\sin.\varphi^n}=\int\frac{d\varphi\cos.\varphi}{\sin.\varphi^n}-\int\frac{d\varphi\cos.\varphi}{\sin.\varphi^{n-2}},$$

quibus reductionibus continuo ulterius progredi licet.

PROBLEMA 28

257. *Formulae* $\frac{d\varphi}{\sin.\varphi^m\cos.\varphi^n}$ *integrale investigare.*

SOLUTIO

Reductiones supra adhibitas huc accommodare licet sumendo in praecedente problemate m negative; ita erit

$$\int\frac{d\varphi}{\sin.\varphi^m\cos.\varphi^n}=+\frac{1}{m+n}\cdot\frac{1}{\sin.\varphi^{m+1}\cos.\varphi^{n-1}}+\frac{m+1}{m+n}\int\frac{d\varphi}{\sin.\varphi^{m+2}\cos.\varphi^n},$$

unde loco m scribendo $m-2$ per conversionem fit

$$\int\frac{d\varphi}{\sin.\varphi^m\cos.\varphi^n}=-\frac{1}{m-1}\cdot\frac{1}{\sin.\varphi^{m-1}\cos.\varphi^{n-1}}+\frac{m+n-2}{m-1}\int\frac{d\varphi}{\sin.\varphi^{m-2}\cos.\varphi^n};$$

altera huic similis est

$$\int\frac{d\varphi}{\sin.\varphi^m\cos.\varphi^n}=\frac{1}{n-1}\cdot\frac{1}{\sin.\varphi^{m-1}\cos.\varphi^{n-1}}+\frac{m+n-2}{n-1}\int\frac{d\varphi}{\sin.\varphi^m\cos.\varphi^{n-2}}.$$

Cum iam in hoc genere formae simplicissimae sint

$$\int\frac{d\varphi}{\sin.\varphi}=l\,\text{tang.}\,\tfrac{1}{2}\varphi,\quad \int\frac{d\varphi}{\cos.\varphi}=l\,\text{tang.}\left(45^\circ+\tfrac{1}{2}\varphi\right),\quad \int\frac{d\varphi}{\sin.\varphi\cos.\varphi}=l\,\text{tang.}\,\varphi,$$

$$\int\frac{d\varphi}{\sin.\varphi^2}=-\cot.\varphi,\quad \int\frac{d\varphi}{\cos.\varphi^2}=\text{tang.}\,\varphi,$$

hinc magis compositas eliciemus

$$\int\frac{d\varphi}{\sin.\varphi\cos.\varphi^2}=\frac{1}{\cos.\varphi}+\int\frac{d\varphi}{\sin.\varphi},$$

$$\int\frac{d\varphi}{\sin.\varphi^2\cos.\varphi}=-\frac{1}{\sin.\varphi}+\int\frac{d\varphi}{\cos.\varphi},$$

$$\int\frac{d\varphi}{\sin.\varphi\cos.\varphi^4}=\frac{1}{3}\cdot\frac{1}{\cos.\varphi^3}+\int\frac{d\varphi}{\sin.\varphi\cos.\varphi^2},$$

$$\int\frac{d\varphi}{\sin.\varphi^4\cos.\varphi}=-\frac{1}{3}\cdot\frac{1}{\sin.\varphi^3}+\int\frac{d\varphi}{\sin.\varphi^2\cos.\varphi},$$

$$\int\frac{d\varphi}{\sin.\varphi\cos.\varphi^6}=\frac{1}{5}\cdot\frac{1}{\cos.\varphi^5}+\int\frac{d\varphi}{\sin.\varphi\cos.\varphi^4},$$

$$\int\frac{d\varphi}{\sin.\varphi^6\cos.\varphi}=-\frac{1}{5}\cdot\frac{1}{\sin.\varphi^5}+\int\frac{d\varphi}{\sin.\varphi^4\cos.\varphi},$$

$$\int\frac{d\varphi}{\sin.\varphi\cos.\varphi^3}=\frac{1}{2}\cdot\frac{1}{\cos.\varphi^2}+\int\frac{d\varphi}{\sin.\varphi\cos.\varphi},$$

$$\int\frac{d\varphi}{\sin.\varphi^3\cos.\varphi}=-\frac{1}{2}\cdot\frac{1}{\sin.\varphi^2}+\int\frac{d\varphi}{\sin.\varphi\cos.\varphi},$$

$$\int\frac{d\varphi}{\sin.\varphi\cos.\varphi^5}=\frac{1}{4}\cdot\frac{1}{\cos.\varphi^4}+\int\frac{d\varphi}{\sin.\varphi\cos.\varphi^3},$$

$$\int\frac{d\varphi}{\sin.\varphi^5\cos.\varphi}=-\frac{1}{4}\cdot\frac{1}{\sin.\varphi^4}+\int\frac{d\varphi}{\sin.\varphi^3\cos.\varphi},$$

$$\int\frac{d\varphi}{\sin.\varphi\cos.\varphi^7}=\frac{1}{6}\cdot\frac{1}{\cos.\varphi^6}+\int\frac{d\varphi}{\sin.\varphi\cos.\varphi^5},$$

$$\int\frac{d\varphi}{\sin.\varphi^7\cos.\varphi}=-\frac{1}{6}\cdot\frac{1}{\sin.\varphi^6}+\int\frac{d\varphi}{\sin.\varphi^5\cos.\varphi}$$

etc.

$$\int \frac{d\varphi}{\sin.\varphi^2 \cos.\varphi^2} = \frac{1}{\sin.\varphi \cos.\varphi} + 2\int \frac{d\varphi}{\sin.\varphi^2} = -\frac{1}{\sin.\varphi \cos.\varphi} + 2\int \frac{d\varphi}{\cos.\varphi^2},$$

$$\int \frac{d\varphi}{\sin.\varphi^2 \cos.\varphi^4} = \frac{1}{3}\cdot\frac{1}{\sin.\varphi \cos\varphi^3} + \frac{4}{3}\int \frac{d\varphi}{\sin.\varphi^2 \cos.\varphi^2},$$

$$\int \frac{d\varphi}{\sin.\varphi^4 \cos.\varphi^2} = -\frac{1}{3}\cdot\frac{1}{\sin.\varphi^3 \cos.\varphi} + \frac{4}{3}\int \frac{d\varphi}{\sin.\varphi^2 \cos.\varphi^2}.$$

Sicque formulae quantumvis compositae ad simpliciores, quarum integratio est in promtu, reducuntur.

COROLLARIUM 1

258. Ambo exponentes ipsius $\sin.\varphi$ et $\cos.\varphi$ simul binario minui possunt; erit enim per priorem reductionem

$$\int \frac{d\varphi}{\sin.\varphi^\mu \cos.\varphi^\nu} = -\frac{1}{\mu-1}\cdot\frac{1}{\sin.\varphi^{\mu-1}\cos.\varphi^{\nu-1}} + \frac{\mu+\nu-2}{\mu-1}\int \frac{d\varphi}{\sin.\varphi^{\mu-2}\cos.\varphi^{\nu}};$$

nunc haec formula per posteriorem ob $m=\mu-2$ et $n=\nu$ dat

$$\int \frac{d\varphi}{\sin.\varphi^{\mu-2}\cos.\varphi^{\nu}} = \frac{1}{\nu-1}\cdot\frac{1}{\sin.\varphi^{\mu-3}\cos.\varphi^{\nu-1}} + \frac{\mu+\nu-4}{\nu-1}\int \frac{d\varphi}{\sin.\varphi^{\mu-2}\cos.\varphi^{\nu-2}},$$

unde concluditur

$$\int \frac{d\varphi}{\sin.\varphi^\mu \cos.\varphi^\nu} = -\frac{1}{\mu-1}\cdot\frac{1}{\sin.\varphi^{\mu-1}\cos.\varphi^{\nu-1}} + \frac{\mu+\nu-2}{(\mu-1)(\nu-1)}\cdot\frac{1}{\sin.\varphi^{\mu-3}\cos.\varphi^{\nu-1}}$$
$$+ \frac{(\mu+\nu-2)(\mu+\nu-4)}{(\mu-1)(\nu-1)}\int \frac{d\varphi}{\sin.\varphi^{\mu-2}\cos.\varphi^{\nu-2}}.$$

COROLLARIUM 2

259. Prioribus membris ad communem denominatorem reductis obtinebitur

$$\int \frac{d\varphi}{\sin.\varphi^\mu \cos.\varphi^\nu}$$
$$= \frac{(\mu-1)\sin.\varphi^2-(\nu-1)\cos.\varphi^2}{(\mu-1)(\nu-1)\sin.\varphi^{\mu-1}\cos.\varphi^{\nu-1}} + \frac{(\mu+\nu-2)(\mu+\nu-4)}{(\mu-1)(\nu-1)}\int \frac{d\varphi}{\sin.\varphi^{\mu-2}\cos.\varphi^{\nu-2}},$$

qua reductione semper ad calculum contrahendum uti licet, nisi vel $\mu=1$ vel $\nu=1$.

SCHOLION

260. Huiusmodi formulae $\frac{d\varphi}{\sin.\varphi^m \cos.\varphi^n}$ etiam hoc modo maxime obvio ad simpliciores reduci possunt, dum numerator per $\sin.\varphi^2 + \cos.\varphi^2 = 1$ multiplicatur, unde fit

$$\int \frac{d\varphi}{\sin.\varphi^m \cos.\varphi^n} = \int \frac{d\varphi}{\sin.\varphi^{m-2} \cos.\varphi^n} + \int \frac{d\varphi}{\sin.\varphi^m \cos.\varphi^{n-2}},$$

quae eousque continuari potest, donec in denominatore unica tantum potestas relinquatur. Ita erit

$$\int \frac{d\varphi}{\sin.\varphi \cos.\varphi} = \int \frac{d\varphi \sin.\varphi}{\cos.\varphi} + \int \frac{d\varphi \cos.\varphi}{\sin.\varphi} = l\frac{\sin.\varphi}{\cos.\varphi},$$

$$\int \frac{d\varphi}{\sin.\varphi^2 \cos.\varphi^2} = \int \frac{d\varphi}{\sin.\varphi^2} + \int \frac{d\varphi}{\cos.\varphi^2} = \frac{\sin.\varphi}{\cos.\varphi} - \frac{\cos.\varphi}{\sin.\varphi}.$$

Quodsi proposita sit haec formula $\int \frac{d\varphi}{\sin.\varphi^n \cos.\varphi^n}$, in subsidium vocari potest esse

$$\sin.\varphi \cos.\varphi = \frac{1}{2}\sin.2\varphi,$$

unde habetur

$$\int \frac{2^n d\varphi}{\sin.2\varphi^n} = 2^{n-1} \int \frac{d\omega}{\sin.\omega^n}$$

posito $\omega = 2\varphi$, quae formula per superiora praecepta resolvitur.

His igitur adminiculis observatis circa formulam $d\varphi \sin.\varphi^m \cos.\varphi^n$, si quidem m et n fuerint numeri integri sive positivi sive negativi, nihil amplius desideratur; sin autem fuerint numeri fracti, nihil admodum praecipiendum occurrit, quandoquidem casus, quibus integratio succedit, quasi sponte se produnt.

Quemadmodum autem integralia, quae exhiberi nequeunt, per series exprimi conveniat, in capite sequente accuratius exponamus.

Nunc vero formulas fractas consideremus, quarum denominator est $a + b\cos.\varphi$ eiusque potestas; tales enim formulae in Theoria Astronomiae frequentissime occurrunt.

PROBLEMA 29

261. *Formulae differentialis* $\frac{d\varphi}{a+b\cos.\varphi}$ *integrale investigare.*

SOLUTIO

Haec investigatio commodius institui nequit, quam ut formula proposita ad formam ordinariam reducatur ponendo $\cos.\varphi = \frac{1-xx}{1+xx}$, ut rationaliter fiat

$$\sin.\varphi = \frac{2x}{1+xx} \quad \text{hincque} \quad d\varphi\cos.\varphi = \frac{2dx(1-xx)}{(1+xx)^2}$$

sicque $d\varphi = \frac{2dx}{1+xx}$. Quia igitur

$$a + b\cos.\varphi = \frac{a+b+(a-b)xx}{1+xx},$$

erit formula nostra

$$\frac{d\varphi}{a+b\cos.\varphi} = \frac{2dx}{a+b+(a-b)xx},$$

quae, prout fuerit $a > b$ vel $a < b$, vel angulum vel logarithmum praebet.

Casu $a > b$ reperitur

$$\int\frac{d\varphi}{a+b\cos.\varphi} = \frac{2}{\sqrt{(aa-bb)}}\,\text{Ang. tang.}\frac{(a-b)x}{\sqrt{(aa-bb)}},$$

casu $a < b$ vero est

$$\int\frac{d\varphi}{a+b\cos.\varphi} = \frac{1}{\sqrt{(bb-aa)}}\,l\frac{\sqrt{(bb-aa)}+x(b-a)}{\sqrt{(bb-aa)}-x(b-a)}.$$

Nunc vero est

$$x = \sqrt{\frac{1-\cos.\varphi}{1+\cos.\varphi}} = \text{tang.}\frac{1}{2}\varphi = \frac{\sin.\varphi}{1+\cos.\varphi};$$

qua restitutione facta cum sit

$$2\,\text{Ang. tang.}\frac{(a-b)x}{\sqrt{(aa-bb)}} = \text{Ang. tang.}\frac{2x\sqrt{(aa-bb)}}{a+b-(a-b)xx}$$

$$= \text{Ang. tang.}\frac{2\sin.\varphi\sqrt{(aa-bb)}}{(a+b)(1+\cos.\varphi)-(a-b)(1-\cos.\varphi)} = \text{Ang. tang.}\frac{\sin.\varphi\sqrt{(aa-bb)}}{a\cos.\varphi+b},$$

quocirca pro casu $a > b$ adipiscimur

$$\int \frac{d\varphi}{a + b \cos.\varphi} = \frac{1}{\sqrt{(aa - bb)}} \text{ Ang. tang. } \frac{\sin.\varphi \sqrt{(aa - bb)}}{a \cos.\varphi + b}$$

seu

$$\int \frac{d\varphi}{a + b \cos.\varphi} = \frac{1}{\sqrt{(aa - bb)}} \text{ Ang. sin. } \frac{\sin.\varphi \sqrt{(aa - bb)}}{a + b \cos.\varphi}$$

sive

$$\int \frac{d\varphi}{a + b \cos.\varphi} = \frac{1}{\sqrt{(aa - bb)}} \text{ Ang. cos. } \frac{a \cos.\varphi + b}{a + b \cos.\varphi}.$$

Pro casu autem $a < b$

$$\int \frac{d\varphi}{a + b \cos.\varphi} = \frac{1}{\sqrt{(bb - aa)}} \, l \frac{\sqrt{(b + a)(1 + \cos.\varphi)} + \sqrt{(b - a)(1 - \cos.\varphi)}}{\sqrt{(b + a)(1 + \cos.\varphi)} - \sqrt{(b - a)(1 - \cos.\varphi)}}$$

seu

$$\int \frac{d\varphi}{a + b \cos.\varphi} = \frac{1}{\sqrt{(bb - aa)}} \, l \frac{a \cos.\varphi + b + \sin.\varphi \sqrt{(bb - aa)}}{a + b \cos.\varphi}.$$

At casu $b = a$ integrale est $= \frac{x}{a} = \frac{1}{a} \text{ tang. } \frac{1}{2}\varphi$, unde fit

$$\int \frac{d\varphi}{1 + \cos.\varphi} = \text{tang. } \frac{1}{2}\varphi = \frac{\sin.\varphi}{1 + \cos.\varphi},$$

quae integralia evanescunt facto $\varphi = 0$.

COROLLARIUM 1

262. Formulae autem $\frac{d\varphi \sin.\varphi}{a + b \cos.\varphi} = \frac{-d.\cos.\varphi}{a + b \cos.\varphi}$ integrale est $\frac{1}{b} \, l \frac{a + b}{a + b \cos.\varphi}$ ita sumtum, ut evanescat posito $\varphi = 0$; sicque habebimus

$$\int \frac{d\varphi \sin.\varphi}{a + b \cos.\varphi} = \frac{1}{b} \, l \frac{a + b}{a + b \cos.\varphi}.$$

COROLLARIUM 2

263. Formula autem $\frac{d\varphi \cos.\varphi}{a + b \cos.\varphi}$ transformatur in $\frac{d\varphi}{b} - \frac{a\,d\varphi}{b(a + b \cos.\varphi)}$, unde integrale per solutionem problematis exhiberi potest

$$\int \frac{d\varphi \cos.\varphi}{a + b \cos.\varphi} = \frac{\varphi}{b} - \frac{a}{b} \int \frac{d\varphi}{a + b \cos.\varphi}.$$

SCHOLION 1

264. Integratione hac inventa etiam huius formulae $\frac{d\varphi}{(a+b\cos.\varphi)^n}$ integrale inveniri potest existente n numero integro; quod fingendo integralis forma commodissime praestari videtur:

$$\int\frac{d\varphi}{(a+b\cos.\varphi)^2}=\frac{A\sin.\varphi}{a+b\cos.\varphi}+m\int\frac{d\varphi}{a+b\cos.\varphi}$$

ac reperitur

$$A=\frac{-b}{aa-bb}\quad\text{et}\quad m=\frac{a}{aa-bb};$$

$$\int\frac{d\varphi}{(a+b\cos.\varphi)^3}=\frac{(A+B\cos.\varphi)\sin.\varphi}{(a+b\cos.\varphi)^2}+m\int\frac{d\varphi}{(a+b\cos.\varphi)^2}$$

reperiturque

$$A=\frac{-b}{aa-bb},\quad B=\frac{-bb}{2a(aa-bb)},\quad m=\frac{2aa+bb}{2a(aa-bb)}$$

similique modo investigatio ad maiores potestates continuari potest, labore quidem non parum taedioso.

Sequenti autem modo negotium facillime expediri videtur. Consideretur scilicet formula generalior $\frac{d\varphi(f+g\cos.\varphi)}{(a+b\cos.\varphi)^{n+1}}$ ac ponatur

$$\int\frac{d\varphi(f+g\cos.\varphi)}{(a+b\cos.\varphi)^{n+1}}=\frac{A\sin.\varphi}{(a+b\cos.\varphi)^n}+\int\frac{d\varphi(B+C\cos.\varphi)}{(a+b\cos.\varphi)^n}$$

sumtisque differentialibus ista prodibit aequatio

$$f+g\cos.\varphi=A\cos.\varphi(a+b\cos.\varphi)+nAb\sin.\varphi^2+(B+C\cos.\varphi)(a+b\cos.\varphi),$$

quae ob $\sin.\varphi^2=1-\cos.\varphi^2$ hanc formam induit

$$\left.\begin{aligned}&-f &&-\ g\cos.\varphi+ &&Ab\cos.\varphi^2\\ &+nAb &&+Aa\cos.\varphi- &&nAb\cos.\varphi^2\\ &+Ba &&+Bb\cos.\varphi+ &&Cb\cos.\varphi^2\\ & &&+Ca\cos.\varphi && \end{aligned}\right\}=0,$$

unde singulis membris nihilo aequatis elicitur

$$A=\frac{ag-bf}{n(aa-bb)},\quad B=\frac{af-bg}{aa-bb}\quad\text{et}\quad C=\frac{(n-1)(ag-bf)}{n(aa-bb)},$$

ita ut haec obtineatur reductio

$$\int\frac{d\varphi(f+g\cos.\varphi)}{(a+b\cos.\varphi)^{n+1}}$$

$$=\frac{(ag-bf)\sin.\varphi}{n(aa-bb)(a+b\cos.\varphi)^n}+\frac{1}{n(aa-bb)}\int\frac{d\varphi(n(af-bg)+(n-1)(ag-bf)\cos.\varphi)}{(a+b\cos.\varphi)^n},$$

cuius ope tandem ad formulam $\int\frac{d\varphi(h+k\cos.\varphi)}{a+b\cos.\varphi}$ pervenitur, cuius integrale

$$=\frac{k}{b}\varphi+\frac{bh-ak}{b}\int\frac{d\varphi}{a+b\cos.\varphi}$$

ex superioribus constat. Perspicuum autem est semper fore $k=0$.

SCHOLION 2

265. Occurrunt etiam eiusmodi formulae, in quas insuper quantitas exponentialis $e^{\alpha\varphi}$ angulum ipsum φ in exponente gerens ingreditur, quas quomodo tractari oporteat, ostendendum videtur, cum hinc methodus reductionum supra exposita maxime illustretur. Hic enim per illam reductionem ad formulam propositae similem pervenitur, unde ipsum integrale colligi poterit. In hunc finem notetur esse $\int e^{\alpha\varphi}d\varphi=\frac{1}{\alpha}e^{\alpha\varphi}$.

PROBLEMA 30

266. *Formulae differentialis* $dy=e^{\alpha\varphi}d\varphi\sin.\varphi^n$ *integrale investigare.*

SOLUTIO

Sumto $e^{\alpha\varphi}d\varphi$ pro factore differentiali erit

$$y=\frac{1}{\alpha}e^{\alpha\varphi}\sin.\varphi^n-\frac{n}{\alpha}\int e^{\alpha\varphi}d\varphi\sin.\varphi^{n-1}\cos.\varphi;$$

simili modo reperitur

$$\int e^{\alpha\varphi}d\varphi\sin.\varphi^{n-1}\cos.\varphi$$

$$=\frac{1}{\alpha}e^{\alpha\varphi}\sin.\varphi^{n-1}\cos.\varphi-\frac{1}{\alpha}\int e^{\alpha\varphi}d\varphi((n-1)\sin.\varphi^{n-2}\cos.\varphi^2-\sin.\varphi^n),$$

quae postrema formula ob $\cos.\varphi^2 = 1 - \sin.\varphi^2$ reducitur ad has

$$(n-1)\int e^{\alpha\varphi} d\varphi \sin.\varphi^{n-2} - n\int e^{\alpha\varphi} d\varphi \sin.\varphi^n,$$

unde habebitur

$$\int e^{\alpha\varphi} d\varphi \sin.\varphi^n = \frac{1}{\alpha} e^{\alpha\varphi} \sin.\varphi^n - \frac{n}{\alpha\alpha} e^{\alpha\varphi} \sin.\varphi^{n-1} \cos.\varphi + \frac{n(n-1)}{\alpha\alpha}\int e^{\alpha\varphi} d\varphi \sin.\varphi^{n-2}$$
$$- \frac{nn}{\alpha\alpha}\int e^{\alpha\varphi} d\varphi \sin.\varphi^n.$$

Quare hanc postremam formulam cum prima coniungendo elicitur

$$\int e^{\alpha\varphi} d\varphi \sin.\varphi^n = \frac{e^{\alpha\varphi} \sin.\varphi^{n-1}(\alpha \sin.\varphi - n\cos.\varphi)}{\alpha\alpha + nn} + \frac{n(n-1)}{\alpha\alpha + nn}\int e^{\alpha\varphi} d\varphi \sin.\varphi^{n-2}.$$

Duobus ergo casibus integrale absolute datur, scilicet $n = 0$ et $n = 1$, eritque

$$\int e^{\alpha\varphi} d\varphi = \frac{1}{\alpha} e^{\alpha\varphi} - \frac{1}{\alpha} \quad \text{et} \quad \int e^{\alpha\varphi} d\varphi \sin.\varphi = \frac{e^{\alpha\varphi}(\alpha\sin.\varphi - \cos.\varphi)}{\alpha\alpha + 1} + \frac{1}{\alpha\alpha + 1}$$

atque ad hos sequentes omnes, ubi n est numerus integer unitate maior, reducuntur.

COROLLARIUM 1

267. Ita si $n = 2$, acquirimus hanc integrationem

$$\int e^{\alpha\varphi} d\varphi \sin.\varphi^2 = \frac{e^{\alpha\varphi} \sin.\varphi(\alpha\sin.\varphi - 2\cos.\varphi)}{\alpha\alpha + 4} + \frac{1\cdot 2}{\alpha(\alpha\alpha + 4)} e^{\alpha\varphi} - \frac{1\cdot 2}{\alpha(\alpha\alpha + 4)};$$

at si sit $n = 3$, istam

$$\int e^{\alpha\varphi} d\varphi \sin.\varphi^3 = \frac{e^{\alpha\varphi} \sin.\varphi^2(\alpha\sin.\varphi - 3\cos.\varphi)}{\alpha\alpha + 9} + \frac{2\cdot 3 e^{\alpha\varphi}(\alpha\sin.\varphi - \cos.\varphi)}{(\alpha\alpha + 1)(\alpha\alpha + 9)}$$
$$+ \frac{2\cdot 3}{(\alpha\alpha + 1)(\alpha\alpha + 9)}$$

integralibus ita sumtis, ut evanescant posito $\varphi = 0$.

COROLLARIUM 2

268. Si igitur determinatis hoc modo integralibus statuatur $\alpha\varphi = -\infty$, ut $e^{\alpha\varphi}$ evanescat, erit in genere

$$\int e^{\alpha\varphi} d\varphi \sin.\varphi^n = \frac{n(n-1)}{\alpha\alpha + nn}\int e^{\alpha\varphi} d\varphi \sin.\varphi^{n-2}$$

hincque integralia pro isto casu $\alpha\varphi = -\infty$ erunt

$$\int e^{\alpha\varphi} d\varphi = -\frac{1}{\alpha}, \qquad \int e^{\alpha\varphi} d\varphi \sin.\varphi = \frac{1}{\alpha\alpha + 1},$$

$$\int e^{\alpha\varphi} d\varphi \sin.\varphi^2 = \frac{-1\cdot 2}{\alpha(\alpha\alpha+4)}, \qquad \int e^{\alpha\varphi} d\varphi \sin.\varphi^3 = \frac{1\cdot 2\cdot 3}{(\alpha\alpha+1)(\alpha\alpha+9)},$$

$$\int e^{\alpha\varphi} d\varphi \sin.\varphi^4 = \frac{-1\cdot 2\cdot 3\cdot 4}{\alpha(\alpha\alpha+4)(\alpha\alpha+16)}, \qquad \int e^{\alpha\varphi} d\varphi \sin.\varphi^5 = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5}{(\alpha\alpha+1)(\alpha\alpha+9)(\alpha\alpha+25)}.$$

COROLLARIUM 3

269. Quare si proponatur haec series infinita

$$s = 1 + \frac{1\cdot 2}{\alpha\alpha+4} + \frac{1\cdot 2\cdot 3\cdot 4}{(\alpha\alpha+4)(\alpha\alpha+16)} + \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}{(\alpha\alpha+4)(\alpha\alpha+16)(\alpha\alpha+36)} + \text{etc.},$$

erit

$$s = -\alpha \int e^{\alpha\varphi} d\varphi (1 + \sin.\varphi^2 + \sin.\varphi^4 + \sin.\varphi^6 + \text{etc.})$$

seu

$$s = -\alpha \int \frac{e^{\alpha\varphi} d\varphi}{\cos.\varphi^2}$$

posito post integrationem $\alpha\varphi = -\infty$.

PROBLEMA 31

270. *Formulae differentialis* $e^{\alpha\varphi} d\varphi \cos.\varphi^n$ *integrale investigare.*

SOLUTIO

Simili modo procedendo ut ante erit

$$\int e^{\alpha\varphi} d\varphi \cos.\varphi^n = \frac{1}{\alpha} e^{\alpha\varphi} \cos.\varphi^n + \frac{n}{\alpha} \int e^{\alpha\varphi} d\varphi \sin.\varphi \cos.\varphi^{n-1},$$

tum vero

$$\int e^{\alpha\varphi} d\varphi \sin.\varphi \cos.\varphi^{n-1}$$

$$= \frac{1}{\alpha} e^{\alpha\varphi} \sin.\varphi \cos.\varphi^{n-1} - \frac{1}{\alpha} \int e^{\alpha\varphi} d\varphi (\cos.\varphi^n - (n-1)\cos.\varphi^{n-2} \sin.\varphi^2),$$

quae postrema formula abit in

$$-(n-1)\int e^{\alpha\varphi}d\varphi\cos.\varphi^{n-2}+n\int e^{\alpha\varphi}d\varphi\cos.\varphi^{n},$$

ita ut sit

$$\int e^{\alpha\varphi}d\varphi\cos.\varphi^{n}=\frac{1}{\alpha}e^{\alpha\varphi}\cos.\varphi^{n}+\frac{n}{\alpha\alpha}e^{\alpha\varphi}\sin.\varphi\cos.\varphi^{n-1}+\frac{n(n-1)}{\alpha\alpha}\int e^{\alpha\varphi}d\varphi\cos.\varphi^{n-2}$$
$$-\frac{nn}{\alpha\alpha}\int e^{\alpha\varphi}d\varphi\cos.\varphi^{n},$$

unde colligimus

$$\int e^{\alpha\varphi}d\varphi\cos.\varphi^{n}=\frac{e^{\alpha\varphi}\cos.\varphi^{n-1}(\alpha\cos.\varphi+n\sin.\varphi)}{\alpha\alpha+nn}+\frac{n(n-1)}{\alpha\alpha+nn}\int e^{\alpha\varphi}d\varphi\cos.\varphi^{n-2};$$

hinc ergo casus simplicissimi sunt

$$\int e^{\alpha\varphi}d\varphi=\frac{1}{\alpha}e^{\alpha\varphi}+C,\quad \int e^{\alpha\varphi}d\varphi\cos.\varphi=\frac{e^{\alpha\varphi}(\alpha\cos.\varphi+\sin.\varphi)}{\alpha\alpha+1}+C,$$

ad quos sequentes omnes, ubi n est numerus integer positivus, reducuntur.

SCHOLION

271. Casibus simplicissimis notatis alia datur via integrale formularum propositarum, quin etiam huius magis patentis $e^{\alpha\varphi}d\varphi\sin.\varphi^{m}\cos.\varphi^{n}$ eruendi. Cum enim productum $\sin.\varphi^{m}\cos.\varphi^{n}$ resolvi possit in aggregatum plurium sinuum vel cosinuum, quorum quisque est huius formae $M\sin.\lambda\varphi$ vel $M\cos.\lambda\varphi$, integratio reducitur ad alterutram harum formularum $e^{\alpha\varphi}d\varphi\sin.\lambda\varphi$ vel $e^{\alpha\varphi}d\varphi\cos.\lambda\varphi$. Ponamus ergo $\lambda\varphi=\omega$, ut habeamus

$$e^{\alpha\varphi}d\varphi\sin.\lambda\varphi=\frac{1}{\lambda}e^{\frac{\alpha}{\lambda}\omega}d\omega\sin.\omega\quad\text{et}\quad e^{\alpha\varphi}d\varphi\cos.\lambda\varphi=\frac{1}{\lambda}e^{\frac{\alpha}{\lambda}\omega}d\omega\cos.\omega,$$

quarum integralia per superiora ita dantur

$$\int e^{\frac{\alpha}{\lambda}\omega}d\omega\sin.\omega=\frac{\lambda e^{\frac{\alpha}{\lambda}\omega}(\alpha\sin.\omega-\lambda\cos.\omega)}{\alpha\alpha+\lambda\lambda}=\frac{\lambda e^{\alpha\varphi}(\alpha\sin.\lambda\varphi-\lambda\cos.\lambda\varphi)}{\alpha\alpha+\lambda\lambda},$$

$$\int e^{\frac{\alpha}{\lambda}\omega}d\omega\cos.\omega=\frac{\lambda e^{\frac{\alpha}{\lambda}\omega}(\alpha\cos.\omega+\lambda\sin.\omega)}{\alpha\alpha+\lambda\lambda}=\frac{\lambda e^{\alpha\varphi}(\alpha\cos.\lambda\varphi+\lambda\sin.\lambda\varphi)}{\alpha\alpha+\lambda\lambda}.$$

Unde tandem colligimus

$$\int e^{\alpha\varphi} d\varphi \sin. \lambda\varphi = \frac{e^{\alpha\varphi}(\alpha \sin. \lambda\varphi - \lambda \cos. \lambda\varphi)}{\alpha\alpha + \lambda\lambda}$$

et

$$\int e^{\alpha\varphi} d\varphi \cos. \lambda\varphi = \frac{e^{\alpha\varphi}(\alpha \cos. \lambda\varphi + \lambda \sin. \lambda\varphi)}{\alpha\alpha + \lambda\lambda}.$$

Si in genere statim loco $\sin. \varphi$ et $\cos. \varphi$ scripsissem $\sin. \lambda\varphi$ et $\cos. \lambda\varphi$, hac reductione non fuisset opus, sed quia hic nihil est difficultatis, brevitati consulendum existimavi.

CAPUT VI

DE EVOLUTIONE INTEGRALIUM PER SERIES SECUNDUM SINUS COSINUSVE ANGULORUM MULTIPLORUM PROGREDIENTES

PROBLEMA 32

272. *Integrale formulae* $\frac{d\varphi}{1+n\cos\varphi}$ *per seriem secundum sinus angulorum multiplorum progredientem exprimere.*

SOLUTIO

Cum sit more consueto per seriem

$$\frac{1}{1+n\cos.\varphi} = 1 - n\cos.\varphi + n^2\cos.\varphi^2 - n^3\cos.\varphi^3 + n^4\cos.\varphi^4 - \text{etc.},$$

potestates cosinus in cosinus angulorum multiplorum convertantur ope formularum in *Introductione*[1]) traditarum ac primo pro potestatibus imparibus:

$$\cos.\varphi = \cos.\varphi,$$

$$\cos.\varphi^3 = \frac{3}{4}\cos.\varphi + \frac{1}{4}\cos.3\varphi,$$

$$\cos.\varphi^5 = \frac{10}{16}\cos.\varphi + \frac{5}{16}\cos.3\varphi + \frac{1}{16}\cos.5\varphi,$$

$$\cos.\varphi^7 = \frac{35}{64}\cos.\varphi + \frac{21}{64}\cos.3\varphi + \frac{7}{64}\cos.5\varphi + \frac{1}{64}\cos.7\varphi,$$

$$\cos.\varphi^9 = \frac{126}{256}\cos.\varphi + \frac{84}{256}\cos.3\varphi + \frac{36}{256}\cos.5\varphi + \frac{9}{256}\cos.7\varphi + \frac{1}{256}\cos.9\varphi,$$

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1) *Introductio*, t. I cap. XIV; vide etiam notam p. 76.

ubi notandum est, si ponatur in genere

$$\cos.\varphi^{2\lambda-1} = A\cos.\varphi + B\cos.3\varphi + C\cos.5\varphi + D\cos.7\varphi + E\cos.9\varphi + \text{etc.},$$

fore

$$A = 2\cdot\frac{1\cdot3\cdot5\cdots(2\lambda-1)}{2\cdot4\cdot6\cdots\cdots2\lambda} = \frac{2}{2^{2\lambda-1}}\cdot\frac{6}{2}\cdot\frac{10}{3}\cdot\frac{14}{4}\cdots\frac{4\lambda-2}{\lambda},$$

$$B = \frac{\lambda-1}{\lambda+1}A,\quad C = \frac{\lambda-2}{\lambda+2}B,\quad D = \frac{\lambda-3}{\lambda+3}C,\quad E = \frac{\lambda-4}{\lambda+4}D \quad \text{etc.}$$

Pro paribus vero potestatibus est

$$\cos.\varphi^0 = 1,$$

$$\cos.\varphi^2 = \frac{1}{2} + \frac{1}{2}\cos.2\varphi,$$

$$\cos.\varphi^4 = \frac{3}{8} + \frac{4}{8}\cos.2\varphi + \frac{1}{8}\cos.4\varphi,$$

$$\cos.\varphi^6 = \frac{10}{32} + \frac{15}{32}\cos.2\varphi + \frac{6}{32}\cos.4\varphi + \frac{1}{32}\cos.6\varphi,$$

$$\cos.\varphi^8 = \frac{35}{128} + \frac{56}{128}\cos.2\varphi + \frac{28}{128}\cos.4\varphi + \frac{8}{128}\cos.6\varphi + \frac{1}{128}\cos.8\varphi.$$

In genere autem si ponatur

$$\cos.\varphi^{2\lambda} = \mathfrak{A} + \mathfrak{B}\cos.2\varphi + \mathfrak{C}\cos.4\varphi + \mathfrak{D}\cos.6\varphi + \mathfrak{E}\cos.8\varphi + \text{etc.},$$

erit

$$\mathfrak{A} = \frac{1\cdot3\cdot5\cdots(2\lambda-1)}{2\cdot4\cdot6\cdots\cdots2\lambda} = \frac{1}{2^{2\lambda-1}}\cdot\frac{6}{2}\cdot\frac{10}{3}\cdot\frac{14}{4}\cdots\frac{4\lambda-2}{\lambda},$$

$$\mathfrak{B} = \frac{2\lambda}{\lambda+1}\mathfrak{A},\quad \mathfrak{C} = \frac{\lambda-1}{\lambda+2}\mathfrak{B},\quad \mathfrak{D} = \frac{\lambda-2}{\lambda+3}\mathfrak{C},\quad \mathfrak{E} = \frac{\lambda-3}{\lambda+4}\mathfrak{D} \quad \text{etc.}$$

Quodsi nunc isti valores substituantur, erit

$$\frac{1}{1+n\cos.\varphi}$$

$$\begin{array}{llllll}
= 1 & -\,n\cos.\varphi & +\frac{1}{2}nn\cos.2\varphi & -\frac{1}{4}n^3\cos.3\varphi & +\frac{1}{8}n^4\cos.4\varphi & -\frac{1}{16}n^5\cos.5\varphi + \text{etc.},\\
+\frac{1}{2}nn & -\frac{3}{4}n^3 & +\frac{4}{8}n^4 & -\frac{5}{16}n^5 & +\frac{6}{32}n^6 & -\frac{7}{64}n^7\\
+\frac{3}{8}n^4 & -\frac{10}{16}n^5 & +\frac{15}{32}n^6 & -\frac{21}{64}n^7 & +\frac{28}{128}n^8 & -\frac{36}{296}n^9\\
+\frac{10}{32}n^6 & -\frac{35}{64}n^7 & +\frac{56}{128}n^8 & -\frac{84}{256}n^9 & & \\
+\frac{35}{128}n^8 & & & & &
\end{array}$$

unde patet, si ponatur

$$\frac{1}{1+n\cos.\varphi} = A - B\cos.\varphi + C\cos.2\varphi - D\cos.3\varphi + E\cos.4\varphi - \text{etc.},$$

esse

$$A = 1 + \frac{1}{2}nn + \frac{3}{8}n^4 + \frac{10}{32}n^6 + \text{etc.},$$

seu

$$A = 1 + \frac{1}{2}n^2 + \frac{1\cdot 3}{2\cdot 4}n^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}n^6 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}n^8 + \text{etc.},$$

sicque evidens est esse

$$A = \frac{1}{\sqrt{(1-nn)}}.$$

Simili modo est

$$B = n + \frac{3}{4}n^3 + \frac{10}{16}n^5 + \text{etc.} = \frac{2}{n}\left(\frac{1}{2}n^2 + \frac{1\cdot 3}{2\cdot 4}n^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}n^6 + \text{etc.}\right)$$

ideoque

$$B = \frac{2}{n}\left(\frac{1}{\sqrt{(1-nn)}} - 1\right).$$

Verum et hunc valorem et sequentes facilius hoc modo definire licet. Cum sit

$$\frac{1}{1+n\cos.\varphi} = A - B\cos.\varphi + C\cos.2\varphi - D\cos.3\varphi + E\cos.4\varphi - \text{etc.},$$

multiplicetur per $1 + n\cos.\varphi$, et quia

$$\cos.\varphi\cos.\lambda\varphi = \frac{1}{2}\cos.(\lambda-1)\varphi + \frac{1}{2}\cos.(\lambda+1)\varphi,$$

fiet

$$\begin{array}{llllll} 1 = & A - & B\cos.\varphi + & C\cos.2\varphi - & D\cos.3\varphi + & E\cos.4\varphi - \text{etc.}, \\ & + An & -\frac{1}{2}Bn & +\frac{1}{2}Cn & -\frac{1}{2}Dn & \\ -\frac{1}{2}Bn + \frac{1}{2}Cn & & -\frac{1}{2}Dn & +\frac{1}{2}En & -\frac{1}{2}Fn & \end{array}$$

unde, quia A iam definivimus, reliqui coefficientes ita determinantur

$$B = \frac{2}{n}(A-1), \qquad C = \frac{2B-2An}{n}, \qquad D = \frac{2C-Bn}{n},$$

$$E = \frac{2D-Cn}{n}, \qquad F = \frac{2E-Dn}{n}, \qquad G = \frac{2F-En}{n}$$

etc.

His igitur coefficientibus inventis integrale facile assignatur; nam cum sit

$$\int d\varphi \cos.\lambda\varphi = \frac{1}{\lambda}\sin.\lambda\varphi,$$

habebimus

$$\int \frac{d\varphi}{1+n\cos.\varphi} = A\varphi - B\sin.\varphi + \frac{1}{2}C\sin.2\varphi - \frac{1}{3}D\sin.3\varphi + \frac{1}{4}E\sin.4\varphi - \text{etc.},$$

quae series secundum sinus angulorum φ, 2φ, 3φ etc. progreditur, uti desiderabatur.

COROLLARIUM 1

273. Primo patet hanc resolutionem locum habere non posse, nisi n sit numerus unitate minor; si enim $n > 1$, singuli coefficientes prodeunt imaginarii. Sin autem sit $n = 1$, ob $1 + \cos.\varphi = 2\cos.\frac{1}{2}\varphi^2$ erit integrale

$$\int \frac{d\varphi}{1+\cos.\varphi} = \int \frac{\frac{1}{2}d\varphi}{\cos.\frac{1}{2}\varphi^2} = \text{tang.}\frac{1}{2}\varphi.$$

COROLLARIUM 2

274. Cum sit

$$A = \frac{1}{\sqrt{(1-nn)}} \quad \text{et} \quad B = \frac{2}{n}\left(\frac{1}{\sqrt{(1-nn)}} - 1\right),$$

reliqui coefficientes C, D, E etc. seriem recurrentem constituunt, ita ut, si bini contigui sint P et Q, sequens futurus sit $\frac{2}{n}Q - P$[1]). Hinc, cum aequationis $zz = \frac{2}{n}z - 1$ radices sint $\frac{1 \pm \sqrt{(1-nn)}}{n}$, quisque terminus in hac forma continetur

$$\alpha\left(\frac{1+\sqrt{(1-nn)}}{n}\right)^{\lambda} + \beta\left(\frac{1-\sqrt{(1-nn)}}{n}\right)^{\lambda}.$$

COROLLARIUM 3

275. Quia autem in nostra lege non A sed $2A$ sumitur, posito $\lambda = 0$ prodire debet $2A$ ideoque

1) Cum sit (§ 272) $C = \frac{2}{n}B - 2A$, series recurrens revera incipit termino D; cf. § 275.
L. S.

$$\alpha + \beta = \frac{2}{\sqrt{(1-nn)}};$$

deinde facto $\lambda = 1$ fieri debet

$$\frac{\alpha+\beta}{n} + \frac{(\alpha-\beta)\sqrt{(1-nn)}}{n} = \frac{2-2\sqrt{(1-nn)}}{n\sqrt{(1-nn)}},$$

unde

$$\alpha - \beta = -\frac{2}{\sqrt{(1-nn)}}.$$

Ergo

$$\alpha = 0 \quad \text{et} \quad \beta = \frac{2}{\sqrt{(1-nn)}}$$

sicque quilibet terminus praeter A erit

$$= \frac{2}{\sqrt{(1-nn)}}\left(\frac{1-\sqrt{(1-nn)}}{n}\right)^{\lambda}.$$

COROLLARIUM 4

276. Coefficientes ergo evoluti ita se habebunt

$$A = \frac{1}{\sqrt{(1-nn)}},$$

$$B = \frac{2-2\sqrt{(1-nn)}}{n\sqrt{(1-nn)}},$$

$$C = \frac{4-2nn-4\sqrt{(1-nn)}}{nn\sqrt{(1-nn)}},$$

$$D = \frac{8-6nn-2(4-nn)\sqrt{(1-nn)}}{n^3\sqrt{(1-nn)}},$$

$$E = \frac{16-16nn+2n^4-2(8-4nn)\sqrt{(1-nn)}}{n^4\sqrt{(1-nn)}},$$

$$F = \frac{32-40nn+10n^4-2(16-12nn+n^4)\sqrt{(1-nn)}}{n^5\sqrt{(1-nn)}},$$

$$G = \frac{64-96nn+36n^4-2n^6-2(32-32nn+6n^4)\sqrt{(1-nn)}}{n^6\sqrt{(1-nn)}}$$

etc.

COROLLARIUM 5

277. Quia $n<1$, hi coefficientes plerumque facilius determinantur per series primum inventas, scilicet

$$A=1+\frac{1}{2}n^2+\frac{1\cdot 3}{2\cdot 4}n^4+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}n^6+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}n^8+\text{etc.},$$

$$B=n\left(1+\frac{3}{4}n^2+\frac{3\cdot 5}{4\cdot 6}n^4+\frac{3\cdot 5\cdot 7}{4\cdot 6\cdot 8}n^6+\frac{3\cdot 5\cdot 7\cdot 9}{4\cdot 6\cdot 8\cdot 10}n^8+\text{etc.}\right),$$

$$C=\frac{1}{2}n^2\left(1+\frac{3\cdot 4}{2\cdot 6}n^2+\frac{3\cdot 4\cdot 5\cdot 6}{2\cdot 6\cdot 4\cdot 8}n^4+\frac{3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}{2\cdot 6\cdot 4\cdot 8\cdot 6\cdot 10}n^6+\text{etc.}\right),$$

$$D=\frac{1}{4}n^3\left(1+\frac{4\cdot 5}{2\cdot 8}n^2+\frac{4\cdot 5\cdot 6\cdot 7}{2\cdot 8\cdot 4\cdot 10}n^4+\frac{4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9}{2\cdot 8\cdot 4\cdot 10\cdot 6\cdot 12}n^6+\text{etc.}\right),$$

$$E=\frac{1}{8}n^4\left(1+\frac{5\cdot 6}{2\cdot 10}n^2+\frac{5\cdot 6\cdot 7\cdot 8}{2\cdot 10\cdot 4\cdot 12}n^4+\frac{5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10}{2\cdot 10\cdot 4\cdot 12\cdot 6\cdot 14}n^6+\text{etc.}\right),$$

$$F=\frac{1}{16}n^5\left(1+\frac{6\cdot 7}{2\cdot 12}n^2+\frac{6\cdot 7\cdot 8\cdot 9}{2\cdot 12\cdot 4\cdot 14}n^4+\frac{6\cdot 7\cdot 8\cdot 9\cdot 10\cdot 11}{2\cdot 12\cdot 4\cdot 14\cdot 6\cdot 16}n^6+\text{etc.}\right)$$

etc.

SCHOLION

278. Cum ex his valoribus sit

$$\int\frac{d\varphi}{1+n\cos.\varphi}=A\varphi-B\sin.\varphi+\frac{1}{2}C\sin.2\varphi-\frac{1}{3}D\sin.3\varphi+\frac{1}{4}E\sin.4\varphi-\text{etc.},$$

in hac serie terminus primus $A\varphi$ imprimis est notandus, quod crescente angulo φ continuo crescat idque in infinitum usque, dum reliqui termini modo crescent modo decrescent; neque tamen certum limitem excedunt, nam $\sin.\lambda\varphi$ neque supra $+1$ crescere neque infra -1 decrescere potest. Cum deinde hoc integrale supra inventum sit

$$\frac{1}{\sqrt{(1-nn)}}\text{Ang. cos.}\frac{n+\cos.\varphi}{1+n\cos.\varphi},$$

series illa huic angulo aequatur. Quare si hic angulus vocetur ω, ut sit

$$d\omega=\frac{d\varphi\sqrt{(1-nn)}}{1+n\cos.\varphi},$$

erit
$$\cos.\omega = \frac{n + \cos.\varphi}{1 + n\cos.\varphi}$$
hincque $n + \cos.\varphi - \cos.\omega - n\cos.\varphi\cos.\omega = 0$, ex quo est vicissim
$$\cos.\varphi = \frac{\cos.\omega - n}{1 - n\cos.\omega};$$
quae formula cum ex illa nascatur sumto n negativo, erit
$$d\varphi = \frac{d\omega\sqrt{(1-nn)}}{1 - n\cos.\omega}$$
et
$$\frac{\varphi}{\sqrt{(1-nn)}} = A\omega + B\sin.\omega + \frac{1}{2}C\sin.2\omega + \frac{1}{3}D\sin.3\omega + \frac{1}{4}E\sin.4\varphi + \text{etc.}$$
Quia vero est
$$\frac{\omega}{\sqrt{(1-nn)}} = A\varphi - B\sin.\varphi + \frac{1}{2}C\sin.2\varphi - \frac{1}{3}D\sin.3\varphi + \frac{1}{4}E\sin.4\varphi - \text{etc.},$$
ob $\frac{1}{\sqrt{(1-nn)}} = A$ habebimus
$$0 = B(\sin.\omega - \sin.\varphi) + \frac{1}{2}C(\sin.2\omega + \sin.2\varphi) + \frac{1}{3}D(\sin.3\omega - \sin.3\varphi) + \text{etc.},$$
cuiusmodi relationes notasse iuvabit.

PROBLEMA 33

279. *Integrale formulae $d\varphi(1 + n\cos.\varphi)^\nu$ per seriem secundum sinus angulorum multiplorum ipsius φ progredientem exprimere.*

SOLUTIO

Cum sit
$$(1 + n\cos.\varphi)^\nu = 1 + \frac{\nu}{1}n\cos.\varphi + \frac{\nu(\nu-1)}{1\cdot 2}n^2\cos.\varphi^2 + \frac{\nu(\nu-1)(\nu-2)}{1\cdot 2\cdot 3}n^3\cos.\varphi^3 + \text{etc.},$$
si ponamus
$$(1 + n\cos.\varphi)^\nu = A + B\cos.\varphi + C\cos.2\varphi + D\cos.3\varphi + E\cos.4\varphi + \text{etc.},$$
erit per formulas supra indicatas

$$A = 1 + \frac{\nu(\nu-1)}{1\cdot 2}\cdot\frac{1}{2}n^2 + \frac{\nu(\nu-1)(\nu-2)(\nu-3)}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{1\cdot 3}{2\cdot 4}n^4$$

$$+ \frac{\nu(\nu-1)\cdots(\nu-5)}{1\cdot 2\cdot 3\cdots 6}\cdot\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}n^6 + \text{etc.},$$

$$B = 2n\left(\frac{\nu}{2} + \frac{\nu(\nu-1)(\nu-2)}{1\cdot 2\cdot 3}\cdot\frac{1\cdot 3}{2\cdot 4}n^2 + \frac{\nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}\cdot\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}n^4 + \text{etc.}\right),$$

quae series ita clarius exhibentur

$$A = 1 + \frac{\nu(\nu-1)}{2\cdot 2}n^2 + \frac{\nu(\nu-1)(\nu-2)(\nu-3)}{2\cdot 2\cdot 4\cdot 4}n^4 + \frac{\nu(\nu-1)\cdots(\nu-5)}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}n^6 + \text{etc.},$$

$$\frac{1}{2}B = \frac{\nu}{2}n + \frac{\nu(\nu-1)(\nu-2)}{2\cdot 2\cdot 4}n^3 + \frac{\nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)}{2\cdot 2\cdot 4\cdot 4\cdot 6}n^5 + \text{etc.}$$

Inventis autem his binis coefficientibus A et B reliqui ex his sequenti modo commodius determinari poterunt. Cum sit

$$\nu l(1 + n\cos.\varphi) = l(A + B\cos.\varphi + C\cos.2\varphi + D\cos.3\varphi + E\cos.4\varphi + \text{etc.}),$$

sumantur differentialia ac per $-d\varphi$ dividendo prodit

$$\frac{\nu n\sin.\varphi}{1+n\cos.\varphi} = \frac{B\sin.\varphi + 2C\sin.2\varphi + 3D\sin.3\varphi + 4E\sin.4\varphi + \text{etc.}}{A + B\cos.\varphi + C\cos.2\varphi + D\cos.3\varphi + E\cos.4\varphi + \text{etc.}}.$$

Iam per crucem multiplicando ob

$$\sin.\lambda\varphi\cos.\varphi = \frac{1}{2}\sin.(\lambda+1)\varphi + \frac{1}{2}\sin.(\lambda-1)\varphi$$

et

$$\sin.\varphi\cos.\lambda\varphi = \frac{1}{2}\sin.(\lambda+1)\varphi - \frac{1}{2}\sin.(\lambda-1)\varphi$$

pervenietur ad hanc aequationem

$$\begin{array}{llllll}
0 = B\sin.\varphi + & 2C\sin.2\varphi + & 3D\sin.3\varphi + & 4E\sin.4\varphi + & 5F\sin.5\varphi + \text{etc.}, \\
 & +\frac{1}{2}Bn & +\frac{2}{2}Cn & +\frac{3}{2}Dn & +\frac{4}{2}En \\
+\frac{2}{2}Cn & +\frac{3}{2}Dn & +\frac{4}{2}En & +\frac{5}{2}Fn & +\frac{6}{2}Gn \\
-\nu An & -\frac{\nu}{2}Bn & -\frac{\nu}{2}Cn & -\frac{\nu}{2}Dn & -\frac{\nu}{2}En \\
+\frac{\nu}{2}Cn & +\frac{\nu}{2}Dn & +\frac{\nu}{2}En & +\frac{\nu}{2}Fn & +\frac{\nu}{2}Gn
\end{array}$$

unde hae sequuntur determinationes

$(\nu+2)Cn+2B-2\nu An=0$	$C=\frac{2\nu An-2B}{(\nu+2)n}$
$(\nu+3)Dn+4C-(\nu-1)Bn=0$	$D=\frac{(\nu-1)Bn-4C}{(\nu+3)n}$
$(\nu+4)En+6D-(\nu-2)Cn=0$	$E=\frac{(\nu-2)Cn-6D}{(\nu+4)n}$
$(\nu+5)Fn+8E-(\nu-3)Dn=0$	$F=\frac{(\nu-3)Dn-8E}{(\nu+5)n}$
$(\nu+6)Gn+10F-(\nu-4)En=0$	$G=\frac{(\nu-4)En-10F}{(\nu+6)n}$

ubi si superiores valores pro A et B substituantur, reperitur

$$C=4n^2\Big(\frac{1\nu(\nu-1)}{2\cdot2\cdot4}+\frac{2\nu(\nu-1)(\nu-2)(\nu-3)}{2\cdot2\cdot4\cdot4\cdot6}n^2+\frac{3\nu(\nu-1)\cdots(\nu-5)}{2\cdot2\cdot4\cdot4\cdot6\cdot6\cdot8}n^4+\text{etc.}\Big),$$

$$D=8n^3\Big(\frac{1\cdot2\nu(\nu-1)(\nu-2)}{2\cdot2\cdot4\cdot4\cdot6}+\frac{2\cdot3\nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)}{2\cdot2\cdot4\cdot4\cdot6\cdot6\cdot8}n^2+\text{etc.}\Big),$$

$$E=16n^4\Big(\frac{1\cdot2\cdot3\nu(\nu-1)(\nu-2)(\nu-3)}{2\cdot2\cdot4\cdot4\cdot6\cdot6\cdot8}+\frac{2\cdot3\cdot4\nu(\nu-1)\cdots(\nu-5)}{2\cdot2\cdot4\cdot4\cdot6\cdot6\cdot8\cdot8\cdot10}n^2+\text{etc.}\Big)$$

etc.,

unde forma sequentium serierum colligitur.

His autem inventis coefficientibus erit integrale quaesitum

$$\int d\varphi(1+n\cos.\varphi)^\nu=A\varphi+B\sin.\varphi+\frac{1}{2}C\sin.2\varphi+\frac{1}{3}D\sin.3\varphi+\frac{1}{4}E\sin.4\varphi+\text{etc.}$$

COROLLARIUM 1

280. Ad similitudinem harum serierum pro C, D, E etc. datarum etiam valor ipsius B ita exprimi potest

$$B=2n\Big(\frac{\nu}{2}+\frac{\nu(\nu-1)(\nu-2)}{2\cdot2\cdot4}n^2+\frac{\nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)}{2\cdot2\cdot4\cdot4\cdot6}n^4+\text{etc.}\Big);$$

series autem pro A inventa formam habet singularem in hac lege non comprehensam.

COROLLARIUM 2

281. Si series A et B inter se comparemus, varias relationes inter eas observare licet, quarum haec primo se offert

$$An + \frac{1}{2}B = \frac{\nu+2}{2}n \left\{ \begin{aligned} &1 + \frac{\nu(\nu-1)}{2\cdot 4}n^2 + \frac{\nu(\nu-1)(\nu-2)(\nu-3)}{2\cdot 4\cdot 4\cdot 6}n^4 \\ &\qquad + \frac{\nu(\nu-1)\cdots(\nu-5)}{2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8}n^6 + \text{etc.} \end{aligned} \right\}$$

quae a serie A tantum secundum denominatores differt.

COROLLARIUM 3

282. Ponamus $\frac{2Ann + Bn}{\nu+2} = N$, ut sit

$$N = n^2 + \frac{\nu(\nu-1)}{2\cdot 4}n^4 + \frac{\nu(\nu-1)(\nu-2)(\nu-3)}{2\cdot 4\cdot 4\cdot 6}n^6 + \text{etc.},$$

$$A = 1 + \frac{\nu(\nu-1)}{2\cdot 2}n^2 + \frac{\nu(\nu-1)(\nu-2)(\nu-3)}{2\cdot 2\cdot 4\cdot 4}n^4 + \text{etc.}$$

Quodsi iam n ut variabilis tractetur, differentiatio praebet

$$\frac{dN}{ndn} = 2 + \frac{\nu(\nu-1)}{2}n^2 + \frac{\nu(\nu-1)(\nu-2)(\nu-3)}{2\cdot 4\cdot 4}n^4 + \text{etc.} = 2A.$$

Cum igitur sit

$$dN = \frac{4Andn + Bdn + 2nndA + ndB}{\nu+2} = 2Andn,$$

erit

$$2\nu Andn = 2nndA + Bdn + ndB.$$

COROLLARIUM 4

283. Ex dato ergo coefficiente A coefficiens B ita per integrationem inveniri potest, ut sit

$$Bn = 2\int(\nu Andn - nndA),$$

vel erit etiam ex illa forma

$$B = \frac{2(\nu+2)}{n}\int Andn - 2An,$$

ubi notandum est posito $n = 0$ integrale $\int Andn$ evanescere debere, quia hoc casu B evanescit.

SCHOLION

284. Series pro litteris B, C, D etc. inventas etiam sequenti modo per continuos factores exprimere licet

$$B=\nu n\left(1+\frac{(\nu-1)(\nu-2)}{2\cdot 4}n^2+\frac{(\nu-3)(\nu-4)}{4\cdot 6}Pn^2+\frac{(\nu-5)(\nu-6)}{6\cdot 8}Pn^2+\text{etc.}\right),$$

$$C=\frac{\nu(\nu-1)}{1\cdot 2}\cdot\frac{n^2}{2}\left(1+\frac{(\nu-2)(\nu-3)}{2\cdot 6}n^2+\frac{(\nu-4)(\nu-5)}{4\cdot 8}Pn^2+\frac{(\nu-6)(\nu-7)}{6\cdot 10}Pn^2+\text{etc.}\right),$$

$$D=\frac{\nu\cdots(\nu-2)}{1\cdots 3}\cdot\frac{n^3}{4}\left(1+\frac{(\nu-3)(\nu-4)}{2\cdot 8}n^2+\frac{(\nu-5)(\nu-6)}{4\cdot 10}Pn^2+\frac{(\nu-7)(\nu-8)}{6\cdot 12}Pn^2+\text{etc.}\right),$$

$$E=\frac{\nu\cdots(\nu-3)}{1\cdots 4}\cdot\frac{n^4}{8}\left(1+\frac{(\nu-4)(\nu-5)}{2\cdot 10}n^2+\frac{(\nu-6)(\nu-7)}{4\cdot 12}Pn^2+\frac{(\nu-8)(\nu-9)}{6\cdot 14}Pn^2+\text{etc.}\right),$$

$$F=\frac{\nu\cdots(\nu-4)}{1\cdots 5}\cdot\frac{n^5}{16}\left(1+\frac{(\nu-5)(\nu-6)}{2\cdot 12}n^2+\frac{(\nu-7)(\nu-8)}{4\cdot 14}Pn^2+\frac{(\nu-9)(\nu-10)}{6\cdot 16}Pn^2+\text{etc.}\right)$$

etc.,

ubi in qualibet serie littera P terminum praecedentem integrum denotat. Atque ope serierum istarum coefficientes plerumque facilius inveniuntur quam ex lege ante tradita, qua quisque ex binis praecedentibus determinatur. Quin etiam haec lex defectu laborat, quod, si ν fuerit numerus integer negativus praeter -1, quidam coefficientes plane non definiantur, quos ergo ex his seriebus desumi oportet. Ita si fuerit $\nu=-2$, erit $B=\nu An=-2An$ et

$$C=\frac{3}{1}\cdot\frac{n^2}{2}\left(1+\frac{4\cdot 5}{2\cdot 6}n^2+\frac{4\cdot 5\cdot 6\cdot 7}{2\cdot 6\cdot 4\cdot 8}n^4+\frac{4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9}{2\cdot 6\cdot 4\cdot 8\cdot 6\cdot 10}n^6+\text{etc.}\right);$$

si sit $\nu=-3$, erit $C=-Bn$ et

$$D=-\frac{4\cdot 5}{1\cdot 2}\cdot\frac{n^3}{4}\left(1+\frac{6\cdot 7}{2\cdot 8}n^2+\frac{6\cdot 7\cdot 8\cdot 9}{2\cdot 8\cdot 4\cdot 10}n^4+\frac{6\cdot 7\cdot 8\cdot 9\cdot 10\cdot 11}{2\cdot 8\cdot 4\cdot 10\cdot 6\cdot 12}n^6+\text{etc.}\right);$$

si sit $\nu=-4$, erit $D=-Cn$ et

$$E=\frac{5\cdot 6\cdot 7}{1\cdot 2\cdot 3}\cdot\frac{n^4}{8}\left(1+\frac{8\cdot 9}{2\cdot 10}n^2+\frac{8\cdot 9\cdot 10\cdot 11}{2\cdot 10\cdot 4\cdot 12}n^4+\frac{8\cdot 9\cdot 10\cdot 11\cdot 12\cdot 13}{2\cdot 10\cdot 4\cdot 12\cdot 6\cdot 14}n^6+\text{etc.}\right);$$

si sit $\nu=-5$, erit $E=-Dn$ et

$$F=-\frac{6\cdot 7\cdot 8\cdot 9}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{n^5}{16}\left(1+\frac{10\cdot 11}{2\cdot 12}n^2+\frac{10\cdot 11\cdot 12\cdot 13}{2\cdot 12\cdot 4\cdot 14}n^4+\frac{10\cdot 11\cdot 12\cdot 13\cdot 14\cdot 15}{2\cdot 12\cdot 4\cdot 14\cdot 6\cdot 16}n^6+\text{etc.}\right)$$

et ita de reliquis.

EXEMPLUM 1

285. *Formulae $d\varphi(1+n\cos.\varphi)^\nu$ integrale evolvere, si ν sit numerus integer positivus.*

Posito

$$(1+n\cos.\varphi)^\nu = A + B\cos.\varphi + C\cos.2\varphi + D\cos.3\varphi + E\cos.4\varphi + \text{etc.}$$

pro singulis valoribus exponentis ν habebimus,

1) si $\nu=1$: $A=1,\quad B=n,\quad C=0$ etc.;

2) si $\nu=2$: $A=1+\frac{1}{2}n^2,\quad B=2n,\quad C=\frac{1}{2}n^2,\quad D=0$ etc.;

3) si $\nu=3$: $A=1+\frac{3}{2}n^2,\quad B=3n\left(1+\frac{1}{4}n^2\right),\quad C=\frac{3}{2}n^2,\quad D=\frac{1}{4}n^3,$

$E=0$ etc.;

4) si $\nu=4$: $A=1+\frac{6}{2}n^2+\frac{3}{8}n^4,\quad B=4n\left(1+\frac{3}{4}n^2\right),\quad C=3n^2\left(1+\frac{1}{6}n^2\right),$

$D=n^3,\quad E=\frac{1}{8}n^4,\quad F=0$ etc.

Hi autem casus nihil habent difficultatis. Ad usum sequentem tantum iuvabit primum terminum absolutum A notasse:

si $\nu=1,\quad A=1,$

si $\nu=2,\quad A=1+\frac{2\cdot1}{2\cdot2}n^2,$

si $\nu=3,\quad A=1+\frac{3\cdot2}{2\cdot2}n^2,$

si $\nu=4,\quad A=1+\frac{4\cdot3}{2\cdot2}n^2+\frac{4\cdot3\cdot2\cdot1}{2\cdot2\cdot4\cdot4}n^4,$

si $\nu=5,\quad A=1+\frac{5\cdot4}{2\cdot2}n^2+\frac{5\cdot4\cdot3\cdot2}{2\cdot2\cdot4\cdot4}n^4,$

si $\nu=6,\quad A=1+\frac{6\cdot5}{2\cdot2}n^2+\frac{6\cdot5\cdot4\cdot3}{2\cdot2\cdot4\cdot4}n^4+\frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{2\cdot2\cdot4\cdot4\cdot6\cdot6}n^6,$

si $\nu=7,\quad A=1+\frac{7\cdot6}{2\cdot2}n^2+\frac{7\cdot6\cdot5\cdot4}{2\cdot2\cdot4\cdot4}n^4+\frac{7\cdot6\cdot5\cdot4\cdot3\cdot2}{2\cdot2\cdot4\cdot4\cdot6\cdot6}n^6$

etc.

EXEMPLUM 2

286. *Formulae* $\frac{d\varphi}{(1+n\cos.\varphi)^{\mu}}$ *integrale per seriem evolvere.*

Posito

$$\frac{1}{(1+n\cos.\varphi)^{\mu}} = A + B\cos.\varphi + C\cos.2\varphi + D\cos.3\varphi + E\cos.4\varphi + \text{etc.}$$

ex praecedentibus formulis ponendo $\nu = -\mu$ erit

$$A = 1 + \frac{\mu(\mu+1)}{2\cdot 2}n^2 + \frac{\mu(\mu+1)(\mu+2)(\mu+3)}{2\cdot 2\cdot 4\cdot 4}n^4 + \frac{\mu(\mu+1)\cdots(\mu+5)}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}n^6 + \text{etc.},$$

$$B = -\mu n\left(1 + \frac{(\mu+1)(\mu+2)}{2\cdot 4}n^2 + \frac{(\mu+3)(\mu+4)}{4\cdot 6}Pn^2 + \frac{(\mu+5)(\mu+6)}{6\cdot 8}Pn^2 + \text{etc.}\right),$$

$$C = \frac{\mu(\mu+1)}{1\cdot 2}\cdot\frac{n^2}{2}\left(1 + \frac{(\mu+2)(\mu+3)}{2\cdot 6}n^2 + \frac{(\mu+4)(\mu+5)}{4\cdot 8}Pn^2 + \frac{(\mu+6)(\mu+7)}{6\cdot 10}Pn^2 + \text{etc.}\right),$$

$$D = -\frac{\mu(\mu+1)(\mu+2)}{1\cdot 2\cdot 3}\cdot\frac{n^3}{4}\left(1 + \frac{(\mu+3)(\mu+4)}{2\cdot 8}n^2 + \frac{(\mu+5)(\mu+6)}{4\cdot 10}Pn^2 + \text{etc.}\right)$$

etc.,

ubi ut ante in quaque serie P terminum praecedentem denotat. Hi autem coefficientes ita a se invicem pendent, ut sit

$$B = \frac{-2(\mu-2)}{n}\int And n - 2An$$

et

$$C = \frac{2B + 2\mu An}{(\mu-2)n}, \quad D = \frac{4C + (\mu+1)Bn}{(\mu-3)n}, \quad E = \frac{6D + (\mu+2)Cn}{(\mu-4)n},$$

$$F = \frac{8E + (\mu+3)Dn}{(\mu-5)n}, \quad G = \frac{10F + (\mu+4)En}{(\mu-6)n}, \quad H = \frac{12G + (\mu+5)Fn}{(\mu-7)n}$$

etc.

Ubi incommodo, quando μ est numerus integer, supra iam remedium est allatum. Hic igitur praecipue investigamus, quomodo coefficientes cuiusque casus ex casu praecedente determinari queant, quod ita fieri poterit. Cum sit

$$\frac{1}{(1+n\cos.\varphi)^{\mu}} = A + B\cos.\varphi + C\cos.2\varphi + D\cos.3\varphi + \text{etc.},$$

ponatur

$$\frac{1}{(1+n\cos.\varphi)^{\mu+1}} = A' + B'\cos.\varphi + C'\cos.2\varphi + D'\cos.3\varphi + \text{etc.};$$

haec igitur series per $1 + n\cos.\varphi$ multiplicata in illam abire debet; est autem productum

$$\begin{array}{llll} A' + & B'\cos.\varphi + & C'\cos.2\varphi + & D'\cos.3\varphi + \text{etc.}, \\ & + A'n & + \frac{1}{2}B'n & + \frac{1}{2}C'n \\ + \frac{1}{2}B'n + \frac{1}{2}C'n & + \frac{1}{2}D'n & + \frac{1}{2}E'n & \end{array}$$

unde colligimus

$$B' = \frac{2(A-A')}{n}, \qquad C' = \frac{2(B-B') - 2A'n}{n},$$

$$D' = \frac{2(C-C') - B'n}{n}, \quad E' = \frac{2(D-D') - C'n}{n} \quad \text{etc.};$$

dummodo ergo coefficiens A' constaret, sequentes B', C', D' etc. haberemus. Videamus igitur, quomodo A' ex A determinari possit; quia est

$$A = 1 + \frac{\mu(\mu+1)}{2\cdot 2}n^2 + \frac{\mu(\mu+1)(\mu+2)(\mu+3)}{2\cdot 2\cdot 4\cdot 4}n^4 + \text{etc.},$$

$$A' = 1 + \frac{(\mu+1)(\mu+2)}{2\cdot 2}n^2 + \frac{(\mu+1)(\mu+2)(\mu+3)(\mu+4)}{2\cdot 2\cdot 4\cdot 4}n^4 + \text{etc.},$$

tractetur n ut variabilis ac prior series per n^μ multiplicata differentietur, ut prodeat

$$\frac{d.An^\mu}{dn} = \mu n^{\mu-1} + \frac{\mu(\mu+1)(\mu+2)}{2\cdot 2}n^{\mu+1} + \frac{\mu(\mu+1)(\mu+2)(\mu+3)(\mu+4)}{2\cdot 2\cdot 4\cdot 4}n^{\mu+3} + \text{etc.},$$

quae series manifesto est $= \mu n^{\mu-1}A'$; quocirca A' ita per A determinatur, ut sit

$$A' = \frac{d.An^\mu}{d.n^\mu} = A + \frac{ndA}{\mu dn}.$$

Cum igitur pro casu $\mu = 1$ invenerimus $A = \frac{1}{\sqrt{(1-nn)}}$, ob $\frac{dA}{dn} = \frac{n}{(1-nn)^{\frac{3}{2}}}$ erit

$$A' = \frac{1}{\sqrt{(1-nn)}} + \frac{nn}{(1-nn)^{\frac{3}{2}}} = \frac{1}{(1-nn)^{\frac{3}{2}}}.$$

Hic iam est valor ipsius A pro $\mu = 2$, unde ob $\frac{dA}{dn} = \frac{3n}{(1-nn)^{\frac{5}{2}}}$ fiet pro $\mu = 3$

$$A = \frac{1}{(1-nn)^{\frac{3}{2}}} + \frac{3nn}{2(1-nn)^{\frac{5}{2}}} = \frac{1+\frac{1}{2}nn}{(1-nn)^{\frac{5}{2}}}.$$

Hoc modo si ulterius progrediamur, reperiemus,

$$\text{si } \mu = 1, \quad A = \frac{1}{\sqrt{(1-nn)}},$$

$$\text{si } \mu = 2, \quad A = \frac{1}{(1-nn)\sqrt{(1-nn)}},$$

$$\text{si } \mu = 3, \quad A = \frac{1+\frac{1}{2}nn}{(1-nn)^2\sqrt{(1-nn)}},$$

$$\text{si } \mu = 4, \quad A = \frac{1+\frac{3}{2}nn}{(1-nn)^3\sqrt{(1-nn)}},$$

$$\text{si } \mu = 5, \quad A = \frac{1+3nn+\frac{3}{8}n^4}{(1-nn)^4\sqrt{(1-nn)}}.$$

COROLLARIUM 1

287. Eodem modo etiam reliqui coefficientes B', C' etc. ex analogis B, C etc. definientur eruntque omnes istae relationes inter se similes, scilicet uti est

$$A' = \frac{d.An^\mu}{d.n^\mu} = A + \frac{ndA}{\mu dn},$$

ita erit

$$B' = \frac{d.Bn^\mu}{d.n^\mu} = B + \frac{ndB}{\mu dn}, \quad C' = \frac{d.Cn^\mu}{d.n^\mu} = C + \frac{ndC}{\mu dn} \quad \text{etc.}$$

COROLLARIUM 2

288. At ante invenimus $B' = \frac{2(A-A')}{n}$, unde fiet

$$B' = -\frac{2dA}{\mu dn} = B + \frac{ndB}{\mu dn}$$

hincque $\mu Bdn + ndB + 2dA = 0$; multiplicetur per $n^{\mu-1}$, ut sit

$$d.Bn^\mu + 2n^{\mu-1}dA = 0,$$

unde integrando

$$Bn^\mu = -2\int n^{\mu-1}dA = -2n^{\mu-1}A + 2(\mu-1)\int An^{\mu-2}dn$$

ideoque

$$B = -\frac{2A}{n} + \frac{2(\mu-1)}{n^\mu}\int An^{\mu-2}dn.$$

At ante [§ 286] habueramus

$$B = -2An - \frac{2(\mu-2)}{n}\int And n.$$

COROLLARIUM 3

289. His valoribus aequatis obtinetur aequatio inter A et n, qua quantitas A per n determinatur; erit enim

$$n^{-\mu}\int n^{\mu-1}dA = An + \frac{(\mu-2)}{n}\int And n,$$

unde per duplicem differentiationem prodit

$$(1-nn)ddA + \frac{dndA}{n} - 2(\mu+1)ndndA - \mu(\mu+1)Adn^2 = 0.$$

SCHOLION 1

290. Si hos valores ipsius A cum superioribus, ubi μ erat numerus integer negativus, inter se comparemus, eximiam convenientiam deprehendemus:

Pro superioribus	Pro his formulis
si $\nu = 0$, $A = 1$	si $\mu = 1$, $A = \frac{1}{\sqrt{(1-nn)}}$
$\nu = 1$, $A = 1$	$\mu = 2$, $A = \frac{1}{(1-nn)\sqrt{(1-nn)}}$
$\nu = 2$, $A = 1 + \frac{1}{2}nn$	$\mu = 3$, $A = \frac{1+\frac{1}{2}nn}{(1-nn)^2\sqrt{(1-nn)}}$
$\nu = 3$, $A = 1 + \frac{3}{2}nn$	$\mu = 4$, $A = \frac{1+\frac{3}{2}nn}{(1-nn)^3\sqrt{(1-nn)}}$
$\nu = 4$, $A = 1 + 3nn + \frac{3}{8}n^4$	$\mu = 5$, $A = \frac{1+3nn+\frac{3}{8}n^4}{(1-nn)^4\sqrt{(1-nn)}}$

etc.,

unde concludimus, si fuerit

$$(1+n\cos.\varphi)^{\nu} = A + B\cos.\varphi + C\cos.2\varphi + \text{etc.},$$
$$(1+n\cos.\varphi)^{-\nu-1} = \mathfrak{A} + \mathfrak{B}\cos.\varphi + \mathfrak{C}\cos.2\varphi + \text{etc.},$$

fore

$$\mathfrak{A} = \frac{A}{(1-nn)^{\nu}\sqrt{(1-nn)}}.$$

Quare cum pro casibus, quibus ν est numerus integer positivus, valor ipsius A facile definiatur, etiam pro casibus, quibus est negativus, inde expedite assignabitur.

SCHOLION 2

291. Cum pro casu $\mu = 1$ supra valores singularum litterarum A, B, C, D etc. sint inventi, scilicet posito brevitatis gratia $\frac{1-\sqrt{(1-nn)}}{n} = m$

$$A = \frac{1}{\sqrt{(1-nn)}}, \quad B = \frac{2m}{\sqrt{(1-nn)}}, \quad C = \frac{2mm}{\sqrt{(1-nn)}}, \quad D = \frac{2m^3}{\sqrt{(1-nn)}}$$

et in genere pro termino quocunque

$$N = \frac{2m^{\lambda}}{\sqrt{(1-nn)}},$$

si pro simili termino casu $\mu = 2$ scribamus N', erit $N' = \frac{d.Nn}{dn}$. Nunc autem est

$$\frac{d.Nn}{dn} = \frac{2m^{\lambda}}{(1-nn)^{\frac{3}{2}}} + \frac{2\lambda n m^{\lambda-1} dm}{dn\sqrt{(1-nn)}},$$

tum vero $\frac{dm}{dn} = \frac{m}{n\sqrt{(1-nn)}}$, unde colligimus

$$N' = \frac{2m^{\lambda}}{(1-nn)^{\frac{3}{2}}} + \frac{2\lambda m^{\lambda}}{1-nn} = \frac{2m^{\lambda}(1+\lambda\sqrt{(1-nn)})}{(1-nn)\sqrt{(1-nn)}}.$$

Quare si statuamus

$$\frac{1}{(1+n\cos.\varphi)^2} = A + B\cos.\varphi + C\cos.2\varphi + D\cos.3\varphi + E\cos.4\varphi + \text{etc.},$$

erit

$$A = \frac{1}{(1-nn)^{\frac{3}{2}}}, \quad B = \frac{2m(1+\sqrt{(1-nn)})}{(1-nn)^{\frac{3}{2}}}, \quad C = \frac{2m^2(1+2\sqrt{(1-nn)})}{(1-nn)^{\frac{3}{2}}},$$

$$D = \frac{2m^3(1+3\sqrt{(1-nn)})}{(1-nn)^{\frac{3}{2}}} \quad \text{etc.}$$

Verum si exponens μ fuerit numerus fractus, coefficientes A, B, C, D, E etc. haud aliter ac per series supra datas definiri posse videntur. Primus autem A modo peculiari vero proxime assignari potest, quemadmodum in problemate sequente docemus.

PROBLEMA 34

292. *Pro evolutione formulae* $(1+n\cos.\varphi)^{\nu}$ *in huiusmodi seriem*

$$A+B\cos.\varphi+C\cos.2\varphi+D\cos.3\varphi+E\cos.4\varphi+\text{etc.}$$

terminum absolutum A *vero proxime definire.*

SOLUTIO

Cum necessario sit $n<1$, series quidem supra inventa pro A convergit, verum si n parum ab unitate deficiat, permultos terminos actu evolvi oportet, antequam valor ipsius A satis exacte prodeat, praecipue si ν fuerit numerus mediocriter magnus tam positivus quam negativus. Quoniam tamen posita evolutione huius formulae

$$(1+n\cos.\varphi)^{-\nu-1}=\mathfrak{A}+\mathfrak{B}\cos.\varphi+\mathfrak{C}\cos.2\varphi+\text{etc.}$$

a termino $\mathfrak{A}$ ille A ita pendet, ut sit $A=(1-nn)^{\nu+\frac{1}{2}}\mathfrak{A}$, pro hoc termino A inveniendo duplicem habebimus seriem

$$A=1+\frac{\nu(\nu-1)}{2\cdot 2}n^2+\frac{\nu(\nu-1)(\nu-2)(\nu-3)}{2\cdot 2\cdot 4\cdot 4}n^4+\frac{\nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)(\nu-5)}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}n^6+\text{etc.},$$

$$A=(1-nn)^{\nu+\frac{1}{2}}\left\{\begin{aligned}&1+\frac{(\nu+1)(\nu+2)}{2\cdot 2}n^2+\frac{(\nu+1)(\nu+2)(\nu+3)(\nu+4)}{2\cdot 2\cdot 4\cdot 4}n^4\\&+\frac{(\nu+1)(\nu+2)(\nu+3)(\nu+4)(\nu+5)(\nu+6)}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}n^6+\text{etc.}\end{aligned}\right\};$$

quovis casu ea usurpari potest, quae magis convergit. Verum tamen quia reliqui coefficientes B, C, D, E etc. tandem convergere debent, hinc alia via ad valorem ipsius A appropinquandi patet.

Quoniam enim hi coefficientes alternatim per pares et impares potestates ipsius n definiuntur, sumto angulo quocunque α erit

$$(1 + n \cos.\alpha)^{\nu} = A + B \cos.\alpha + C \cos.2\alpha + D \cos.3\alpha + E \cos.4\alpha + \text{etc.}$$

et

$$(1 - n \cos.\alpha)^{\nu} = A - B \cos.\alpha + C \cos.2\alpha - D \cos.3\alpha + E \cos.4\alpha - \text{etc.}$$

His igitur additis prodit

$$\frac{1}{2}(1 + n \cos.\alpha)^{\nu} + \frac{1}{2}(1 - n \cos.\alpha)^{\nu} = A + C \cos.2\alpha + E \cos.4\alpha + G \cos.6\alpha + \text{etc.};$$

ubi si pro α scribamus $90^{0} - \alpha$, erit

$$\frac{1}{2}(1 + n \sin.\alpha)^{\nu} + \frac{1}{2}(1 - n \sin.\alpha)^{\nu} = A - C \cos.2\alpha + E \cos.4\alpha - G \cos.6\alpha + \text{etc.},$$

unde his additis semissis terminorum denuo tollitur. Formemus plures huiusmodi expressiones ac ponamus brevitatis gratia

$$\frac{1}{4}(1 + n \cos.\alpha)^{\nu} + \frac{1}{4}(1 - n \cos.\alpha)^{\nu} + \frac{1}{4}(1 + n \sin.\alpha)^{\nu} + \frac{1}{4}(1 - n \sin.\alpha)^{\nu} = \mathfrak{A},$$

$$\frac{1}{4}(1 + n \cos.\beta)^{\nu} + \frac{1}{4}(1 - n \cos.\beta)^{\nu} + \frac{1}{4}(1 + n \sin.\beta)^{\nu} + \frac{1}{4}(1 - n \sin.\beta)^{\nu} = \mathfrak{B},$$

$$\frac{1}{4}(1 + n \cos.\gamma)^{\nu} + \frac{1}{4}(1 - n \cos.\gamma)^{\nu} + \frac{1}{4}(1 + n \sin.\gamma)^{\nu} + \frac{1}{4}(1 - n \sin.\gamma)^{\nu} = \mathfrak{C}$$

etc.

et pro coefficientibus B, C, D, E etc. scribamus respective (1), (2), (3), (4) etc., quo facilius terminos ab initio quantumvis remotos repraesentare possimus. Habebimus ergo

$$\mathfrak{A} = A + (4) \cos.4\alpha + (8) \cos.8\alpha + (12) \cos.12\alpha + \text{etc.},$$

$$\mathfrak{B} = A + (4) \cos.4\beta + (8) \cos.8\beta + (12) \cos.12\beta + \text{etc.},$$

$$\mathfrak{C} = A + (4) \cos.4\gamma + (8) \cos.8\gamma + (12) \cos.12\gamma + \text{etc.}$$

etc.

Atque hinc sequentes approximationes adipiscimur.

I. Si capiamus $4\alpha = \frac{\pi}{2}$ seu $\alpha = \frac{\pi}{8}$, prodit

$$\mathfrak{A} = A - (8) + (16) - (24) + \text{etc.}$$

Ergo

$$A = \mathfrak{A} + (8) - (16) + (24) - \text{etc.}$$

Quare si termini (8) et sequentes ob parvitatem contemni queant, erit satis exacte $A = \mathfrak{A}$.

II. Sumamus duas series ac statuamus $4\alpha = \frac{\pi}{4}$ et $4\beta = \frac{3\pi}{4}$, ut sit $\alpha = \frac{\pi}{16}$ et $\beta = \frac{3\pi}{16}$; erit

$$\cos.4\alpha + \cos.4\beta = 0, \quad \cos.8\alpha + \cos.8\beta = 0, \quad \cos.12\alpha + \cos.12\beta = 0$$

et

$$\cos.16\alpha + \cos.16\beta = -2,$$

unde sequitur

$$\mathfrak{A} + \mathfrak{B} = 2A - 2(16) + 2(32) - 2(48) + \text{etc.}$$

ideoque

$$A = \frac{1}{2}(\mathfrak{A} + \mathfrak{B}) + (16) - (32) + \text{etc.},$$

ubi numeri (16), (32) plerumque tam erunt parvi, ut negligi queant.

III. Addamus tres series ac statuamus $4\alpha = \frac{\pi}{6}$, $4\beta = \frac{3\pi}{6}$, $4\gamma = \frac{5\pi}{6}$, ut sit $\alpha = \frac{\pi}{24}$, $\beta = \frac{\pi}{8}$, $\gamma = \frac{5\pi}{24}$, eritque

$$\begin{aligned} &\cos.4\alpha + \cos.4\beta + \cos.4\gamma = 0, \qquad &&\cos.16\alpha + \cos.16\beta + \cos.16\gamma = 0,\\ &\cos.8\alpha + \cos.8\beta + \cos.8\gamma = 0, &&\cos.20\alpha + \cos.20\beta + \cos.20\gamma = 0,\\ &\cos.12\alpha + \cos.12\beta + \cos.12\gamma = 0, &&\cos.24\alpha + \cos.24\beta + \cos.24\gamma = -3,\end{aligned}$$

unde colligitur

$$A = \frac{1}{3}(\mathfrak{A} + \mathfrak{B} + \mathfrak{C}) + (24) - (48) + \text{etc.}$$

IV. Si haec determinatio non satis exacta videatur, addantur quatuor eiusmodi expressiones $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ sitque $4\alpha = \frac{\pi}{8}$, $4\beta = \frac{3\pi}{8}$, $4\gamma = \frac{5\pi}{8}$, $4\delta = \frac{7\pi}{8}$

ac reperietur

$$\mathfrak{A}+\mathfrak{B}+\mathfrak{C}+\mathfrak{D}=4A-4(32)+4(64)-\text{etc.},$$

ergo multo propius

$$A=\frac{1}{4}(\mathfrak{A}+\mathfrak{B}+\mathfrak{C}+\mathfrak{D}).$$

COROLLARIUM 1

293. Ex invento autem valore A sequens B satis expedite reperitur, cum sit

$$B=\frac{2(\nu+2)}{n}\int And n-2An.$$

Quatenus ergo in A ingreditur membrum $(1\pm n\cos.\alpha)^{\nu}$ vel $(1+nf)^{\nu}$, dum f omnes illos sinus et cosinus complectitur, inde pro B oritur

$$\frac{2(\nu+2)}{n}\int ndn(1+nf)^{\nu}-2n(1+nf)^{\nu}=\frac{2-2(1-\nu nf)(1+nf)^{\nu}}{(\nu+1)nff}.$$

COROLLARIUM 2

294. Cognitis autem coefficientibus A et B quemadmodum sequentes omnes ex illis derivari possint, supra ostendimus. Iis vero inventis integratio formulae $d\varphi(1+n\cos.\varphi)^{\nu}$ per se est manifesta.

PROBLEMA 35

295. *Integrale formulae $d\varphi l(1+n\cos.\varphi)$ per seriem secundum sinus angulorum φ, 2φ, 3φ etc. progredientem evolvere.*

SOLUTIO

Cum sit

$$l(1+n\cos.\varphi)=n\cos.\varphi-\frac{1}{2}n^2\cos.\varphi^2+\frac{1}{3}n^3\cos.\varphi^3-\frac{1}{4}n^4\cos.\varphi^4+\text{etc.},$$

erit his potestatibus ad simplices cosinus reductis

$$l(1+n\cos.\varphi) = n\cos.\varphi - \frac{1}{2}\cdot\frac{1}{2}n^2\cos.2\varphi + \frac{1}{3}\cdot\frac{1}{4}n^3\cos.3\varphi - \frac{1}{4}\cdot\frac{1}{8}n^4\cos.4\varphi + \text{etc.}$$

$$-\frac{1}{2}\cdot\frac{1}{2}n^2 + \frac{1}{3}\cdot\frac{3}{4}n^3 \qquad -\frac{1}{4}\cdot\frac{4}{8}n^4 \qquad +\frac{1}{5}\cdot\frac{5}{16}n^5$$

$$-\frac{1}{4}\cdot\frac{3}{8}n^4 + \frac{1}{5}\cdot\frac{10}{16}n^5 \qquad -\frac{1}{6}\cdot\frac{15}{32}n^6$$

$$-\frac{1}{6}\cdot\frac{10}{32}n^6 + \frac{1}{7}\cdot\frac{35}{64}n^7$$

$$-\frac{1}{8}\cdot\frac{35}{128}n^8$$

Quare si ponamus

$$l(1+n\cos.\varphi) = -A + B\cos.\varphi - C\cos.2\varphi + D\cos.3\varphi - \text{etc.},$$

erit

$$A = \frac{1}{2}\cdot\frac{n^2}{2} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{n^4}{4} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{n^6}{6} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{n^8}{8} + \text{etc.};$$

considerato ergo numero n ut variabili erit

$$\frac{n\,dA}{dn} = \frac{1}{2}nn + \frac{1\cdot 3}{2\cdot 4}n^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}n^6 + \text{etc.} = \frac{1}{\sqrt{(1-nn)}} - 1.$$

Hinc

$$dA = \frac{dn}{n\sqrt{(1-nn)}} - \frac{dn}{n},$$

unde integratio praebet

$$A = l\,\frac{1-\sqrt{(1-nn)}}{n} - ln + C = l\,\frac{2-2\sqrt{(1-nn)}}{nn};$$

hoc enim modo evanescente n fit $A = l1 = 0$.

Tum vero erit

$$\frac{1}{2}B = \frac{1}{2}n + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{n^3}{3} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{n^5}{5} + \text{etc.},$$

unde differentiatio praebet

$$\frac{nn\,dB}{2\,dn} = \frac{1}{2}nn + \frac{1\cdot 3}{2\cdot 4}n^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}n^6 + \text{etc.} = \frac{1}{\sqrt{(1-nn)}} - 1,$$

ergo

$$\frac{1}{2}dB = \frac{dn}{nn\sqrt{(1-nn)}} - \frac{dn}{nn}$$

et integrando

$$\frac{1}{2}B = \frac{-\sqrt{(1-nn)}}{n} + \frac{1}{n} + C = \frac{1-\sqrt{(1-nn)}}{n}$$

integrali ita determinato, ut evanescat posito $n = 0$. Quocirca pro binis primis terminis habemus

$$A = l\,\frac{2-2\sqrt{(1-nn)}}{nn} \quad \text{et} \quad B = \frac{2-2\sqrt{(1-nn)}}{n},$$

ut sit $A = l\,\frac{B}{n}$. At pro reliquis differentiemus aequationem assumtam

$$\frac{-n\,d\varphi \sin.\varphi}{1+n\cos.\varphi} = -Bd\varphi\sin.\varphi + 2Cd\varphi\sin.2\varphi - 3Dd\varphi\sin.3\varphi + 4Ed\varphi\sin.4\varphi - \text{etc.}$$

seu

$$0 = \frac{n\sin.\varphi}{1+n\cos.\varphi} - B\sin.\varphi + 2C\sin.2\varphi - 3D\sin.3\varphi + 4E\sin.4\varphi - \text{etc.}$$

Quare per $2 + 2n\cos.\varphi$ multiplicando prodit

$$\begin{array}{llllll} 0 = 2n\sin.\varphi & -\,2B\sin.\varphi & +\,4C\sin.2\varphi & -\,6D\sin.3\varphi & +\,8E\sin.4\varphi & -\text{ etc.,} \\ & & -\,Bn & +\,2Cn & -\,3Dn & \\ & +\,2Cn & -\,3Dn & +\,4En & -\,5Fn & \end{array}$$

unde colligimus

$$C = \frac{B-n}{n}, \quad D = \frac{4C-Bn}{3n}, \quad E = \frac{6D-2Cn}{4n}, \quad F = \frac{8E-3Dn}{5n}.$$

Cum igitur sit $B = \frac{2-2\sqrt{(1-nn)}}{n}$, erit

$$C = \frac{2-nn-2\sqrt{(1-nn)}}{nn} \quad \text{seu} \quad C = \left(\frac{1-\sqrt{(1-nn)}}{n}\right)^2,$$

tum vero

$$D = \frac{2}{3}\left(\frac{1-\sqrt{(1-nn)}}{n}\right)^3, \quad E = \frac{2}{4}\left(\frac{1-\sqrt{(1-nn)}}{n}\right)^4, \quad F = \frac{2}{5}\left(\frac{1-\sqrt{(1-nn)}}{n}\right)^5 \text{ etc.}$$

Hinc, si brevitatis gratia ponamus $\frac{1-\sqrt{(1-nn)}}{n} = m$, erit

$$l(1+n\cos.\varphi) = -l\,\frac{2m}{n} + \frac{2}{1}m\cos.\varphi - \frac{2}{2}m^2\cos.2\varphi + \frac{2}{3}m^3\cos.3\varphi - \frac{2}{4}m^4\cos.4\varphi + \text{etc.}$$

ideoque integrale quaesitum

$$\int d\varphi\, l(1+n\cos.\varphi) = \text{Const.} - \varphi\, l\frac{2m}{n} + \frac{2}{1}m\sin.\varphi - \frac{2}{4}m^2\sin.2\varphi$$

$$+\frac{2}{9}m^3\sin.3\varphi - \frac{2}{16}m^4\sin.4\varphi + \frac{2}{25}m^5\sin.5\varphi - \text{etc.}$$

COROLLARIUM

296. Quodsi ergo ponamus $n = 1$, erit $m = 1$ et

$$l(1+\cos.\varphi) = -l2 + \frac{2}{1}\cos.\varphi - \frac{2}{2}\cos.2\varphi + \frac{2}{3}\cos.3\varphi - \frac{2}{4}\cos.4\varphi + \text{etc.}$$

et

$$l(1-\cos.\varphi) = -l2 - \frac{2}{1}\cos.\varphi - \frac{2}{2}\cos.2\varphi - \frac{2}{3}\cos.3\varphi - \frac{2}{4}\cos.4\varphi - \text{etc.}$$

Cum iam sit

$$1+\cos.\varphi = 2\cos.\frac{1}{2}\varphi^2 \quad \text{et} \quad 1-\cos.\varphi = 2\sin.\frac{1}{2}\varphi^2,$$

erit

$$l\cos.\frac{1}{2}\varphi = -l2 + \cos.\varphi - \frac{1}{2}\cos.2\varphi + \frac{1}{3}\cos.3\varphi - \frac{1}{4}\cos.4\varphi + \text{etc.}$$

et

$$l\sin.\frac{1}{2}\varphi = -l2 - \cos.\varphi - \frac{1}{2}\cos.2\varphi - \frac{1}{3}\cos.3\varphi - \frac{1}{4}\cos.4\varphi - \text{etc.},$$

hinc

$$l\,\text{tang.}\frac{1}{2}\varphi = -2\cos.\varphi - \frac{2}{3}\cos.3\varphi - \frac{2}{5}\cos.5\varphi - \frac{2}{7}\cos.7\varphi - \text{etc.}$$

CAPUT VII

METHODUS GENERALIS INTEGRALIA QUAECUNQUE PROXIME INVENIENDI

PROBLEMA 36

297. *Formulae integralis cuiuscunque* $y = \int X dx$ *valorem vero proxime indagare.*

SOLUTIO

Cum omnis formula integralis per se sit indeterminata, ea semper ita determinari solet, ut, si variabili x certus quidam valor, puta a, tribuatur, ipsum integrale $y = \int X dx$ datum valorem, puta b, obtineat. Integratione igitur hoc modo determinata quaestio huc redit, ut, si variabili x alius quicunque valor ab a diversus tribuatur, valor, quem tum integrale y sit habiturum, definiatur. Tribuamus ergo ipsi x primo valorem parum ab a discrepantem, puta $x = a + \alpha$, ut α sit quantitas valde parva, et quia functio X parum variatur, sive pro x scribatur a sive $a + \alpha$, eam tanquam constantem spectare licebit. Hinc ergo formulae differentialis $X dx$ integrale erit $Xx + \text{Const.} = y$; sed quia posito $x = a$ fieri debet $y = b$ et valor ipsius X quasi manet immutatus, erit $Xa + \text{Const.} = b$ ideoque $\text{Const.} = b - Xa$, unde consequimur $y = b + X(x - a)$. Quare si ipsi x valorem $a + \alpha$ tribuamus, habebimus valorem convenientem ipsius y, qui sit $= b + \beta$; ac iam simili modo ex hoc casu definire poterimus y, si ipsi x tribuatur alius valor parum superans $a + \alpha$; posito igitur $a + \alpha$ loco x valor ipsius X inde ortus denuo pro constante haberi poterit indeque fiet $y = b + \beta + X(x - a - \alpha)$. Hanc igitur operationem continuare licet, quousque lubuerit; cuius ratio quo melius perspiciatur, rem ita repraesentemus:

$$\begin{aligned}
&\text{si}\quad x=a, &&\text{fiat}\quad X=A &&\text{et}\quad y=b,\\
&\text{si}\quad x=a', &&\text{fiat}\quad X=A' &&\text{et}\quad y=b'=b+A(a'-a),\\
&\text{si}\quad x=a'', &&\text{fiat}\quad X=A'' &&\text{et}\quad y=b''=b'+A'(a''-a'),\\
&\text{si}\quad x=a''', &&\text{fiat}\quad X=A''' &&\text{et}\quad y=b'''=b''+A''(a'''-a'')
\end{aligned}$$

etc.,

ubi valores a, a', a'', a''' etc. secundum differentias valde parvas procedere ponuntur. Erit ergo $b'=b+A(a'-a)$, quippe in quam abit formula inventa $y=b+X(x-a)$; fit enim $X=A$, quia ponitur $x=a$; tum vero tribuitur ipsi x valor $=a'$, cui respondet $y=b'$; simili modo erit $b''=b'+A'(a''-a')$, tum $b'''=b''+A''(a'''-a'')$ etc., uti supra posuimus. Restituendo ergo valores praecedentes habebimus

$$\begin{aligned}
b' &= b+A(a'-a),\\
b'' &= b+A(a'-a)+A'(a''-a'),\\
b''' &= b+A(a'-a)+A'(a''-a')+A''(a'''-a''),\\
b'''' &= b+A(a'-a)+A'(a''-a')+A''(a'''-a'')+A'''(a''''-a''')
\end{aligned}$$

etc.,

unde, si x quantumvis excedet a, series a', a'', a''' etc. crescendo continuetur ad x et ultimum aggregatum dabit valorem ipsius y.

COROLLARIUM 1

298. Si incrementa, quibus x augetur, aequalia statuantur, scilicet α, ut sit $a'=a+\alpha$, $a''=a+2\alpha$, $a'''=a+3\alpha$ etc., quibus valoribus pro x substitutis functio X abeat in A', A'', A''' etc., atque ultimus illorum valorum, puta $a+n\alpha$, sit $=x$, horum vero X, erit

$$y=b+\alpha(A+A'+A''+A'''+\cdots+X).$$

COROLLARIUM 2

299. Valor ergo integralis y per summationem seriei A, A', A'', $\cdots X$, cuius termini ex formula X formantur ponendo loco x successive a, $a+\alpha$,

$a + 2\alpha, \cdots a + n\alpha$, eruitur. Summa enim illius seriei per differentiam α multiplicata et ad b adiecta dabit valorem ipsius y, qui ipsi $x = a + n\alpha$ respondet.

COROLLARIUM 3

300. Quo minores statuuntur differentiae, secundum quas valor ipsius x increscat, eo accuratius hoc modo valor ipsius y definitur, siquidem termini seriei A, A', A'' etc. inde etiam secundum parvas differentias progrediantur; nisi enim hoc eveniat, illa determinatio nimis erit incerta.

COROLLARIUM 4

301. Haec ergo approximatio ex doctrina serierum ita explicatur.

Ex indicibus

$$a, \ a', \ a'', \ a''', \ \cdots \ x$$

formetur series

$$A, \ A', \ A'', \ A''', \ \cdots \ X,$$

cuius ergo terminus generalis X ex formula differentiali $dy = Xdx$ datur. Tum in hac serie sit terminus ultimum praecedens $'X$, respondens indici $'x$ hincque nova formetur series

$$A(a' - a), \quad A'(a'' - a'), \quad A''(a''' - a''), \ \cdots \ 'X(x - 'x);$$

cuius summa si ponatur $= S$, erit integrale $y = \int Xdx = b + S$ proxime.

SCHOLION 1

302. Hoc modo integratio vulgo explicari solet, ut dicatur esse summatio omnium valorum formulae differentialis Xdx, si variabili x successive omnes valores a dato quodam a usque ad x tribuantur, qui secundum differentiam dx procedunt, hanc differentiam autem infinite parvam accipi oportere. Similis igitur haec ratio integrationem repraesentandi est illi, qua in Geometria lineae ut aggregata infinitorum punctorum concipi solent; quae idea quemadmodum, si rite explicetur, admitti potest, ita etiam illa integrationis explicatio tolerari potest, dummodo ad vera principia, uti hic fecimus, revocetur, ut omni cavillationi occurratur. Ex methodo igitur exposita utique patet integrationem per summationem vero proxime obtineri posse neque vero

exacte expediri, nisi differentiae infinite parvae, hoc est nullae, statuantur. Atque ex hoc fonte tam nomen integrationis, quae etiam summatio vocari solet, quam signum integralis $\int$ est natum, quae re bene explicata omnino retineri possunt.

SCHOLION 2

303. Si pro singulis intervallis, in quae saltum ab a ad x distinximus, quantitates A, A', A'', A''' etc. revera essent constantes, integrale $\int X dx$ accurate impetraremus. Eatenus ergo error inest, quatenus pro singulis illis intervallis istae quantitates non sunt constantes. Ac pro primo quidem intervallo, quo variabilis x a termino a ad a' procedit, A est valor ipsius X termino a conveniens, alteri autem termino a' respondet A'; unde, quatenus non est $A' = A$, eatenus error irrepit. Cum igitur in istius intervalli initio sit $X = A$, in fine autem $X = A'$, conveniret potius medium quoddam inter A et A' assumi, id quod in correctione huius methodi mox tradenda observabitur. Interim hic notasse iuvabit pari iure pro quovis intervallo valorem tam finalem quam initialem capi posse, ubi simul hoc perspicitur,[1]) si altero modo in excessu peccetur, altero plerumque in defectu errari. Ex quo hinc binas expressiones eruere licet, quarum altera valorem ipsius y nimis magnum, altera nimis parvum sit praebitura, ita ut illae quasi limites veri valoris ipsius y constituant. Quemadmodum ergo rem repraesentavimus § 301, valor ipsius $y = \int X dx$ intra hos duos limites continebitur

$$b + A(a' - a) + A'(a'' - a') + A''(a''' - a'') + \cdots + 'X(x - 'x)$$

et

$$b + A'(a' - a) + A''(a'' - a') + A'''(a''' - a'') + \cdots + X(x - 'x),$$

quibus cognitis ad veritatem propius accedere licet.

SCHOLION 3

304. Iam notavimus intervalla illa, per quae x successive increscere assumimus, ideo valde parva statui debere, ut valores respondentes A, A', A'' etc. parum a se invicem discrepent; atque hinc potissimum iudicari oportet, utrum illa intervalla $a' - a$, $a'' - a'$, $a''' - a''$ etc. inter se aequalia

1) Quae sequuntur non valent, nisi crescente x functio X aut continuo crescit aut continuo decrescit. L. S.

an inaequalia capi conveniat. Ubi enim valor ipsius X mutando x parum mutatur, ibi intervalla, per quae x procedit, tuto maiora constitui possunt; ubi autem evenit, ut ipsi x levi mutatione inducta functio X vehementer varietur, ibi intervalla minima accipi debent. Veluti si sit $X = \frac{1}{\sqrt{(1-xx)}}$, perspicuum est, ubi x proxime ad unitatem accedit, quantumvis parvum intervallum, per quod x augeatur, accipiatur, functionem X maximam mutationem pati posse, quia tandem sumto $x = 1$ ea adeo in infinitum excrescit. His igitur casibus ista approximatione pro eo saltem intervallo, in cuius altero termino X fit infinita, uti non licet; sed huic incommodo facile remedium affertur, dum formula ope idoneae substitutionis in aliam transformatur vel dum pro hoc saltem intervallo peculiaris integratio instituitur. Veluti si proposita sit formula $\frac{x\,dx}{\sqrt{(1-x^3)}}$, pro intervallo ab $x = 1 - \omega$ ad $x = 1$ illa methodo integrale non reperitur, at posito $x = 1 - z$, quia termini ipsius z sunt 0 et ω, erit z quantitas minima, unde formula erit $\frac{dz(1-z)}{\sqrt{(3z-3z^2+z^3)}} = \frac{dz}{\sqrt{3z}}$, cuius integrale $\frac{2\sqrt{z}}{\sqrt{3}}$ pro intervallo illo praebet partem integralis $\frac{2\sqrt{\omega}}{\sqrt{3}}$. Quod artificium in omnibus huiusmodi casibus adhiberi potest; ipsam autem methodum descriptam aliquot exemplis illustrari opus est.

EXEMPLUM 1

305. *Integrale* $y = \int x^n dx$ *ita sumtum, ut evanescat posito* $x = 0$, *proxime exhibere.*

Hic est $a = 0$ et $b = 0$, tum $X = x^n$; iam valores ipsius x a 0 crescant per communem differentiam α, ut sint

$$\begin{array}{llllllll} \text{indices} & 0, & \alpha, & 2\alpha, & 3\alpha, & 4\alpha, & \cdots & x, \\ \text{series} & 0, & \alpha^n, & 2^n\alpha^n, & 3^n\alpha^n, & 4^n\alpha^n, & \cdots & x^n, \end{array}$$

et terminus ultimum praecedens est $(x-\alpha)^n$, quare integralis

$$y = \int x^n dx = \frac{1}{n+1} x^{n+1}$$

limites sunt

$$\alpha(0 + \alpha^n + 2^n\alpha^n + 3^n\alpha^n + \cdots + (x-\alpha)^n)$$

et

$$\alpha(\alpha^n + 2^n\alpha^n + 3^n\alpha^n + \cdots + x^n),$$

qui eo erunt arctiores, quo minus intervallum α accipiatur. Ita si $\alpha = 1$, erunt limites

$$0 + 1 + 2^n + 3^n + 4^n \cdots + (x-1)^n$$

et

$$1 + 2^n + 3^n + 4^n + \cdots + x^n;$$

si sumatur $\alpha = \frac{1}{2}$, erunt limites

$$\frac{1}{2^{n+1}}(0 + 1 + 2^n + 3^n + 4^n + \cdots + (2x-1)^n)$$

et

$$\frac{1}{2^{n+1}}(1 + 2^n + 3^n + 4^n + \cdots + (2x)^n);$$

ac si in genere sit $\alpha = \frac{1}{m}$, erunt limites

$$\frac{1}{m^{n+1}}(0 + 1 + 2^n + 3^n + 4^n + \cdots + (mx-1)^n)$$

et

$$\frac{1}{m^{n+1}}(1 + 2^n + 3^n + 4^n + \cdots + (mx)^n),$$

quorum hic illum superat excessu $\frac{x^n}{m}$, unde patet, si numerus m sumatur infinitus, utrumque limitem verum integralis $y = \frac{1}{n+1}x^{n+1}$ esse praebiturum valorem.

COROLLARIUM 1

306. Seriei ergo $1 + 2^n + 3^n + 4^n + \cdots + (mx)^n$ summa eo propius ad $\frac{1}{n+1}(mx)^{n+1}$ accedit, quo maior capiatur numerus m; quare posito $mx = z$ huius progressionis

$$1 + 2^n + 3^n + 4^n + \cdots + z^n$$

summa eo propius ad $\frac{1}{n+1}z^{n+1}$ accedit, quo maior fuerit numerus z.

COROLLARIUM 2

307. Ex priore autem limite posito $mx = z$ eadem quantitas $\frac{1}{n+1}z^{n+1}$ proxime exhibet summam huius seriei

$$0 + 1 + 2^n + 3^n + 4^n + \cdots + (z-1)^n,$$

unde medium sumendo erit accuratius

$$1 + 2^n + 3^n + 4^n + \cdots + (z-1)^n + \frac{1}{2} z^n = \frac{1}{n+1} z^{n+1}$$

seu addendo utrinque $\frac{1}{2} z^n$ habebimus

$$1 + 2^n + 3^n + 4^n + \cdots + z^n = \frac{1}{n+1} z^{n+1} + \frac{1}{2} z^n \text{ proxime},$$

quod congruit cum iis, quae de vera huius progressionis summa sunt cognita.

EXEMPLUM 2

308. *Integrale* $y = \int \frac{dx}{x^n}$ *ita sumtum, ut evanescat posito* $x = 1$, *proxime exhibere.*

Erit ergo $a = 1$ et $b = 0$, unde, si ab a ad x intervallum progressionis statuatur $= \alpha$, erunt

$$\text{indices} \quad a, \quad a+\alpha, \quad a+2\alpha, \quad a+3\alpha, \cdots x$$

et

$$\text{termini seriei} \quad \frac{1}{a^n}, \quad \frac{1}{(a+\alpha)^n}, \quad \frac{1}{(a+2\alpha)^n}, \quad \frac{1}{(a+3\alpha)^n}, \cdots \frac{1}{x^n} = X,$$

ubi terminus ultimum praecedens est $\frac{1}{(x-\alpha)^n} = {}'X$. Cum nunc nostrum integrale sit

$$y = \frac{1}{n-1} - \frac{1}{(n-1)x^{n-1}},$$

eius valor intra hos limites continebitur

$$\alpha \left(1 + \frac{1}{(1+\alpha)^n} + \frac{1}{(1+2\alpha)^n} + \frac{1}{(1+3\alpha)^n} + \cdots + \frac{1}{(x-\alpha)^n}\right)$$

et

$$\alpha \left(\frac{1}{(1+\alpha)^n} + \frac{1}{(1+2\alpha)^n} + \frac{1}{(1+3\alpha)^n} + \cdots + \frac{1}{x^n}\right).$$

Quare posito $\alpha = \frac{1}{m}$ erunt hi limites

$$m^{n-1} \left(\frac{1}{m^n} + \frac{1}{(m+1)^n} + \frac{1}{(m+2)^n} + \frac{1}{(m+3)^n} + \cdots + \frac{1}{(mx-1)^n}\right)$$

et

$$m^{n-1} \left(\frac{1}{(m+1)^n} + \frac{1}{(m+2)^n} + \frac{1}{(m+3)^n} + \frac{1}{(m+4)^n} + \cdots + \frac{1}{(mx)^n}\right),$$

qui, quo maior accipiatur numerus m, eo propius ad valorem integralis $\frac{1}{n-1}-\frac{1}{(n-1)x^{n-1}}$ accedunt. Notandum autem est casu $n=1$ integrale fore $=lx$.

COROLLARIUM 1

309. Quodsi ponamus $mx=m+z$, ut sit $x=\frac{m+z}{m}$, prodibunt hae progressiones

$$m^{n-1}\left(\frac{1}{m^n}+\frac{1}{(m+1)^n}+\frac{1}{(m+2)^n}+\cdots+\frac{1}{(m+z-1)^n}\right)$$

et

$$m^{n-1}\left(\frac{1}{(m+1)^n}+\frac{1}{(m+2)^n}+\frac{1}{(m+3)^n}+\cdots+\frac{1}{(m+z)^n}\right),$$

quarum summa alterius maior est, alterius minor quam

$$\frac{1}{n-1}-\frac{m^{n-1}}{(n-1)(m+z)^{n-1}}=\frac{(m+z)^{n-1}-m^{n-1}}{(n-1)(m+z)^{n-1}};$$

casu autem $n=1$ haec expressio abit in $l\left(1+\frac{z}{m}\right)$.

COROLLARIUM 2

310. Cum prior progressio maior sit quam posterior, erit

$$\frac{1}{m^n}+\frac{1}{(m+1)^n}+\frac{1}{(m+2)^n}+\cdots+\frac{1}{(m+z-1)^n}>\frac{(m+z)^{n-1}-m^{n-1}}{(n-1)m^{n-1}(m+z)^{n-1}},$$

$$\frac{1}{(m+1)^n}+\frac{1}{(m+2)^n}+\frac{1}{(m+3)^n}+\cdots+\frac{1}{(m+z)^n}<\frac{(m+z)^{n-1}-m^{n-1}}{(n-1)m^{n-1}(m+z)^{n-1}};$$

addatur hic utrinque $\frac{1}{m^n}$, ibi vero $\frac{1}{(m+z)^n}$ et sumatur medium arithmeticum; erit exactius

$$\frac{1}{m^n}+\frac{1}{(m+1)^n}+\frac{1}{(m+2)^n}+\frac{1}{(m+3)^n}+\cdots+\frac{1}{(m+z)^n}$$
$$=\frac{(2m+n-1)(m+z)^n-(2z+2m-n+1)m^n}{2(n-1)m^n(m+z)^n},$$

quae expressio casu $n=1$ abit in $l\left(1+\frac{z}{m}\right)+\frac{1}{2m}+\frac{1}{2(m+z)}$.

COROLLARIUM 3

311. Ponatur $z = mv$ et habebimus sequentis seriei summam proxime expressam

$$\frac{1}{m^n} + \frac{1}{(m+1)^n} + \frac{1}{(m+2)^n} + \cdots + \frac{1}{m^n(1+v)^n} = \frac{(2m+n-1)(1+v)^n - 2m(1+v) + n - 1}{2(n-1)m^n(1+v)^n}$$

et casu $n = 1$

$$\frac{1}{m} + \frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{m+mv} = l(1+v) + \frac{2+v}{2m(1+v)},$$

unde, si $v = 1$, erit proxime

$$\frac{1}{m^n} + \frac{1}{(m+1)^n} + \frac{1}{(m+2)^n} + \cdots + \frac{1}{2^n m^n} = \frac{2^n(2m+n-1) - 4m + n - 1}{2^{n+1}(n-1)m^n}$$

et

$$\frac{1}{m} + \frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{2m} = l2 + \frac{3}{4m}.$$

COROLLARIUM 4

312. Hinc nascitur regula logarithmos quantumvis magnorum numerorum proxime assignandi, dum series vulgares tantum pro numeris parum ab unitate differentibus valent. Scribamus enim u pro $1 + v$ et habebimus

$$lu = \frac{1}{m} + \frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{mu} - \frac{1+u}{2mu},$$

unde lu eo accuratius definitur, quo maior sumatur numerus m.

EXEMPLUM 3

313. *Integrale* $y = \int \frac{cdx}{cc+xx}$ *ita sumtum, ut evanescat posito* $x = 0$, *proxime exprimere.*

Hoc integrale, ut novimus, est $y = \text{Ang. tang.} \frac{x}{c}$, ad quem valorem proxime exhibendum est $a = 0$ et $b = 0$; si ergo valor ipsius x ab 0 per differentiam constantem α crescere statuatur, ob $X = \frac{c}{cc+xx}$ erunt eius valores

pro indicibus	$0,$	$\alpha,$	$2\alpha,$	$\cdots$	x
series	$\frac{1}{c},$	$\frac{c}{cc+\alpha\alpha},$	$\frac{c}{cc+4\alpha\alpha},$	$\cdots$	$\frac{c}{cc+xx},$

cuius terminus ultimum praecedens est $'X = \frac{c}{cc+(x-\alpha)^2}$. Quare integralis nostri $y = \text{Ang. tang.} \frac{x}{c}$ valor proxime est

$$\alpha\left(\frac{1}{c} + \frac{c}{cc+\alpha\alpha} + \frac{c}{cc+4\alpha\alpha} + \cdots + \frac{c}{cc+(x-\alpha)^2}\right),$$

alter vero proxime minor, quia hic est nimis magnus, est

$$\alpha\left(\frac{c}{cc+\alpha\alpha} + \frac{c}{cc+4\alpha\alpha} + \frac{c}{cc+9\alpha\alpha} + \cdots + \frac{c}{cc+xx}\right).$$

Inter quos si medium capiatur, ibi $\alpha \cdot \frac{1}{c}$, hic vero $\alpha \cdot \frac{c}{cc+xx}$ adiiciendo propius erit

$$\alpha\left(\frac{c}{cc} + \frac{c}{cc+\alpha\alpha} + \frac{c}{cc+4\alpha\alpha} + \frac{c}{cc+9\alpha\alpha} + \cdots + \frac{c}{cc+xx}\right)$$

$$= \text{Ang. tang.} \frac{x}{c} + \frac{\alpha}{2}\left(\frac{1}{c} + \frac{c}{cc+xx}\right) = \text{Ang. tang.} \frac{x}{c} + \frac{\alpha(2cc+xx)}{2c(cc+xx)}.$$

Pro hoc ergo angulo valorem proxime verum habemus

$$\text{Ang. tang.} \frac{x}{c} = \alpha c\left(\frac{1}{cc} + \frac{1}{cc+\alpha\alpha} + \frac{1}{cc+4\alpha\alpha} + \cdots + \frac{1}{cc+xx}\right) - \frac{\alpha(2cc+xx)}{2c(cc+xx)},$$

qui eo minus a veritate discrepabit, quo minor fuerit α numerus ratione ipsius c. Quodsi ergo pro c numerum valde magnum sumamus, pro α unitatem accipere licet, unde posito $x = cv$ erit

$$\text{Ang. tang.}\, v = c\left(\frac{1}{cc} + \frac{1}{cc+1} + \frac{1}{cc+4} + \frac{1}{cc+9} + \cdots + \frac{1}{cc+ccvv}\right) - \frac{2+vv}{2c(1+vv)}$$

idque eo exactius, quo maior capiatur numerus c.

COROLLARIUM 1

314. Si ponamus $c = 1$, quo casu error insignis esse debet, fiet

$$\text{Ang. tang.}\, v = 1 + \frac{1}{1+1} + \frac{1}{1+4} + \frac{1}{1+9} + \cdots + \frac{1}{1+vv} - \frac{2+vv}{2(1+vv)}.$$

Sit $v=1$; erit Ang. tang. $1=\frac{\pi}{4}=1+\frac{1}{2}-\frac{3}{4}=\frac{3}{4}$ hincque $\pi=3$, quod non multum abhorret a vero.

Si ponamus $c=2$, prodit

$$\text{Ang. tang.}\, v=2\left(\frac{1}{4}+\frac{1}{4+1}+\frac{1}{4+4}+\frac{1}{4+9}+\cdots+\frac{1}{4+4vv}\right)-\frac{2+vv}{4(1+vv)},$$

unde, si $v=1$, colligitur Ang. tang. $1=\frac{\pi}{4}=2\left(\frac{1}{4}+\frac{1}{4+1}+\frac{1}{4+4}\right)-\frac{3}{8}=\frac{23}{20}-\frac{3}{8}=\frac{31}{40}$ sicque $\pi=\frac{31}{10}=3{,}1$ propius accedens.

COROLLARIUM 2

315. Sit $c=6$ eritque

$$\text{Ang. tang.}\, v=6\left(\frac{1}{36}+\frac{1}{36+1}+\frac{1}{36+4}+\cdots+\frac{1}{36+36vv}\right)-\frac{2+vv}{12(1+vv)},$$

unde, si $v=\frac{1}{2}$ et $v=\frac{1}{3}$, oritur

$$\text{Ang. tang.}\,\frac{1}{2}=6\left(\frac{1}{36}+\frac{1}{36+1}+\frac{1}{36+4}+\frac{1}{36+9}\right)-\frac{3}{20},$$

$$\text{Ang. tang.}\,\frac{1}{3}=6\left(\frac{1}{36}+\frac{1}{36+1}+\frac{1}{36+4}\right)-\frac{19}{120}.$$

At est Ang. tang. $\frac{1}{2}$ + Ang. tang. $\frac{1}{3}$ = Ang. tang. $1=\frac{\pi}{4}$. Ergo

$$\frac{\pi}{4}=12\left(\frac{1}{36}+\frac{1}{37}+\frac{1}{40}\right)+\frac{2}{15}-\frac{37}{120}=\frac{1063}{1110}-\frac{7}{40}=\frac{695}{888}$$

seu $\pi=\frac{695}{222}=3{,}1306$.

COROLLARIUM 3

316. Sin autem ibi statim ponamus $v=1$, erit

$$\frac{\pi}{4}=6\left(\frac{1}{36}+\frac{1}{37}+\frac{1}{40}+\frac{1}{45}+\frac{1}{52}+\frac{1}{61}+\frac{1}{72}\right)-\frac{1}{8},$$

unde fit $\pi=3{,}13696$ multo propius veritati; plurium scilicet terminorum additio propius ad veritatem perducit.

PROBLEMA 37

317. *Methodum ad integralium valores appropinquandi ante expositam perfectiorem reddere, ut minus a veritate aberretur.*

SOLUTIO

Sit $y = \int X dx$ formula integralis proposita, cuius valorem iam constet esse $y = b$, si ponatur $x = a$, sive is sit datus per ipsam integrationis conditionem sive iam per aliquot operationes inde derivatus; ac tribuamus iam ipsi x valorem parum superantem illum a, cui respondet $y = b$, tum vero fiat $X = A$, si ponatur $x = a$. In superiori autem methodo assumsimus, dum x parum supra a excrescit, manere X constantem $= A$ ideoque fore

$$\int X dx = A(x-a).$$

At quatenus X non est constans, eatenus non est $\int X dx = X(x-a)$, sed revera habetur

$$\int X dx = X(x-a) - \int (x-a) dX.$$

Ponamus igitur $dX = P dx$ eritque

$$\int (x-a) dX = \int P(x-a) dx,$$

et si iam $P = \frac{dX}{dx}$, quamdiu x non multum a excedit, ut constantem spectemus, habebimus

$$\int P(x-a) dx = \frac{1}{2} P(x-a)^2$$

sicque fiet

$$y = \int X dx = b + X(x-a) - \frac{1}{2} P(x-a)^2,$$

qui valor iam propius ad veritatem accedit, etsi pro X et P ii valores capiantur, quos induunt vel posito $x = a$ vel posito $x = a + \alpha$, maiore scilicet valore, ad quem hac operatione x crescere statuimus; ex quo hinc, prout vel $x = a$ vel $x = a + \alpha$ ponimus, geminos limites obtinebimus, inter quos veritas subsistit. Simili autem modo ulterius progredi poterimus; cum enim P non sit constans, erit

$$\int P(x-a)dx = \frac{1}{2}P(x-a)^2 - \frac{1}{2}\int (x-a)^2 dP,$$

unde, si statuamus $dP = Qdx$, erit

$$\int (x-a)^2 dP = \int Q(x-a)^2 dx = \frac{1}{3}Q(x-a)^3,$$

siquidem Q ut quantitatem constantem spectemus, ita ut sit

$$y = \int Xdx = b + X(x-a) - \frac{1}{2}P(x-a)^2 + \frac{1}{2\cdot 3}Q(x-a)^3.$$

Eadem ergo methodo si ulterius procedamus, ponendo

$$X = \frac{dy}{dx}, \quad P = \frac{dX}{dx}, \quad Q = \frac{dP}{dx}, \quad R = \frac{dQ}{dx}, \quad S = \frac{dR}{dx} \quad \text{etc.}$$

invenimus

$$y = b + X(x-a) - \frac{1}{2}P(x-a)^2 + \frac{1}{2\cdot 3}Q(x-a)^3 - \frac{1}{2\cdot 3\cdot 4}R(x-a)^4$$
$$+ \frac{1}{2\cdot 3\cdot 4\cdot 5}S(x-a)^5 - \text{etc.},$$

quae series vehementer convergit, si modo x non multum superet a, atque adeo, si in infinitum continuetur, verum valorem ipsius y exhibebit, siquidem in functionibus X, P, Q, R etc. valor extremus $x = a + \alpha$ substituatur. Nisi autem eam seriem in infinitum extendere velimus, praestabit per intervalla procedere tribuendo ipsi x successive valores a, a', a'', a''', a'''' etc. ac tum pro singulis valores litteris X, P, Q, R, S etc. convenientes quaeri oportet, qui sint, ut sequuntur: Si fuerit

$$x = a, \quad a', \quad a'', \quad a''', \quad a^{\text{IV}}, \quad a^{\text{V}} \quad \text{etc.},$$

fiat

$$X = A, \quad A', \quad A'', \quad A''', \quad A^{\text{IV}}, \quad A^{\text{V}} \quad \text{etc.},$$
$$\frac{dX}{dx} = P = B, \quad B', \quad B'', \quad B''', \quad B^{\text{IV}}, \quad B^{\text{V}} \quad \text{etc.},$$
$$\frac{dP}{dx} = Q = C, \quad C', \quad C'', \quad C''', \quad C^{\text{IV}}, \quad C^{\text{V}} \quad \text{etc.},$$
$$\frac{dQ}{dx} = R = D, \quad D', \quad D'', \quad D''', \quad D^{\text{IV}}, \quad D^{\text{V}} \quad \text{etc.}$$
$$\text{etc.};$$

tum vero sit

$$y = b, \quad b', \quad b'', \quad b''', \quad b^{\text{IV}}, \quad b^{\text{V}} \quad \text{etc.},$$

quibus constitutis erit, ut ex antecedentibus colligere licet,

$$b' = b + A'(a' - a) - \frac{1}{2}B'(a' - a)^2 + \frac{1}{6}C'(a' - a)^3 - \frac{1}{24}D'(a' - a)^4 + \text{etc.},$$

$$b'' = b' + A''(a'' - a') - \frac{1}{2}B''(a'' - a')^2 + \frac{1}{6}C''(a'' - a')^3 - \frac{1}{24}D''(a'' - a')^4 + \text{etc.},$$

$$b''' = b'' + A'''(a''' - a'') - \frac{1}{2}B'''(a''' - a'')^2 + \frac{1}{6}C'''(a''' - a'')^3 - \frac{1}{24}D'''(a''' - a'')^4 + \text{etc.},$$

$$b^{\text{IV}} = b''' + A^{\text{IV}}(a^{\text{IV}} - a''') - \frac{1}{2}B^{\text{IV}}(a^{\text{IV}} - a''')^2 + \frac{1}{6}C^{\text{IV}}(a^{\text{IV}} - a''')^3 - \frac{1}{24}D^{\text{IV}}(a^{\text{IV}} - a''')^4 + \text{etc.}$$

etc.,

quae expressiones eousque continuentur, donec pro valore ipsius x quantumvis ab initiali a discrepante valor ipsius y obtineatur.

COROLLARIUM 1

318. Haec igitur approximandi methodus eo utitur theoremate, cuius veritas iam in *Calculo Differentiali*[1]) est demonstrata: quodsi y eiusmodi fuerit functio ipsius x, quae posito $x = a$ fiat $= b$, ac statuatur

$$\frac{dy}{dx} = X, \quad \frac{dX}{dx} = P, \quad \frac{dP}{dx} = Q, \quad \frac{dQ}{dx} = R \quad \text{etc.},$$

fore generaliter

$$y = b + X(x - a) - \frac{1}{2}P(x - a)^2 + \frac{1}{6}Q(x - a)^3 - \frac{1}{24}R(x - a)^4 + \frac{1}{120}S(x - a)^5 - \text{etc.}$$

1) *Institutiones calculi differentialis*, partis posterioris Cap. III; vide etiam notam p. 41. L. S.

COROLLARIUM 2

319. Si hanc seriem in infinitum continuare vellemus, non opus esset valorem ipsius x parum tantum ab a diversum assumere. Verum quo ista series magis convergens reddatur, expedit saltum ab a ad x in intervalla dispesci et pro singulis operationem hic descriptam institui.

COROLLARIUM 3

320. Si valores ipsius x ab a per differentias constantes $= \alpha$ crescere faciamus sitque ultimus $a + n\alpha = x$, ita ut, si fuerit

$$x = a, \quad a+\alpha, \quad a+2\alpha, \quad a+3\alpha, \quad \cdots \quad x,$$

fiat

$$X = A, \quad A', \quad A'', \quad A''', \quad \cdots \quad X,$$

$$\frac{dX}{dx} = P = B, \quad B', \quad B'', \quad B''', \quad \cdots \quad P,$$

$$\frac{dP}{dx} = Q = C, \quad C', \quad C'', \quad C''', \quad \cdots \quad Q,$$

$$\frac{dQ}{dx} = R = D, \quad D', \quad D'', \quad D''', \quad \cdots \quad R$$

etc.

indeque

$$y = b, \quad b', \quad b'', \quad b''', \quad \cdots \quad y,$$

erit pro valore $x = x$ omnes series colligendo

$$\begin{aligned} y = b + {} & \alpha(A' + A'' + A''' + \cdots + X) \\ & - \frac{1}{2}\alpha^2(B' + B'' + B''' + \cdots + P) \\ & + \frac{1}{6}\alpha^3(C' + C'' + C''' + \cdots + Q) \\ & - \frac{1}{24}\alpha^4(D' + D'' + D''' + \cdots + R) \end{aligned}$$

etc.

SCHOLION 1

321. Demonstratio theorematis Corollario 1 memorati, cui haec methodus approximandi innititur, ex natura differentialium ita instruitur. Sit y functio ipsius x, quae posito $x=a$ fiat $y=b$, et quaeramus valorem ipsius y, si x utcunque excedat a. Incipiamus a valore ipsius maximo, qui est x, et per differentialia descendamus atque ex differentialibus patet,

si fuerit x	fore y
$x-dx$	$y-dy+ddy-d^3y+d^4y-\text{etc.}$
$x-2dx$	$y-2dy+3ddy-4d^3y+5d^4y-\text{etc.}$
$x-3dx$	$y-3dy+6ddy-10d^3y+15d^4y-\text{etc.}$
$\vdots$	$\vdots$
$x-ndx$	$y-ndy+\frac{n(n+1)}{1\cdot 2}ddy-\frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}d^3y$ $+\frac{n(n+1)(n+2)(n+3)}{1\cdot 2\cdot 3\cdot 4}d^4y-\text{etc.}$

Ponamus nunc $x-ndx=a$; erit $n=\frac{x-a}{dx}$ ideoque numerus infinitus; tum vero valor pro y resultans per hypothesin esse debet $=b$, quamobrem habebimus

$$b=y-\frac{(x-a)dy}{dx}+\frac{(x-a)^2ddy}{1\cdot 2dx^2}-\frac{(x-a)^3d^3y}{1\cdot 2\cdot 3dx^3}+\frac{(x-a)^4d^4y}{1\cdot 2\cdot 3\cdot 4dx^4}-\text{etc.}$$

Quodsi iam statuamus

$$\frac{dy}{dx}=X,\quad \frac{dX}{dx}=P,\quad \frac{dP}{dx}=Q,\quad \frac{dQ}{dx}=R\ \text{etc.},$$

reperimus ut ante

$$y=b+X(x-a)-\frac{1}{2}P(x-a)^2+\frac{1}{6}Q(x-a)^3-\frac{1}{24}R(x-a)^4+\text{etc.}$$

Unde patet, si x quam minime superet a, sufficere statui $y=b+X(x-a)$, quod est fundamentum approximationis primum propositae, illius scilicet limitis, quo X ex valore maiore ipsius x definitur.

SCHOLION 2

322. Quemadmodum hoc ratiocinium nobis alterum tantum limitem supra assignatum patefecit, ita ad alterum limitem hoc ratiocinium nos manuducet. Scilicet uti ante ab x ad a descendimus, ita nunc ab a ad x ascendamus;

si abeat a in	tum b abibit in
$a+da$	$b+db$
$a+2da$	$b+2db+ddb$
$a+3da$	$b+3db+3ddb+d^3b$
$\vdots$	$\vdots$
$a+nda$	$b+ndb+\frac{n(n-1)}{1\cdot 2}ddb+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}d^3b+$ etc.

Sit iam $a+nda=x$ seu $n=\frac{x-a}{da}$ et valor ipsius b fiet $=y$. Sint autem A, B, C, D etc. valores superiorum functionum X, P, Q, R etc., si loco x scribatur a, eritque pro praesenti casu

$$A=\frac{db}{da},\quad B=\frac{ddb}{da^2},\quad C=\frac{d^3b}{da^3}\quad \text{etc.}$$

Quocirca habebimus

$$y=b+A(x-a)+\frac{1}{2}B(x-a)^2+\frac{1}{6}C(x-a)^3+\frac{1}{24}D(x-a)^4+\text{etc.},$$

quae series superiori praeter signa omnino est similis; ac si x parum excedat a, ut $b+A(x-a)$ satis exacte valorem ipsius y indicet, hinc alter limes supra assignatus nascitur. Quodsi autem progressum ab a ad x ut supra § 320 in intervalla aequalia secundum differentiam α dispescamus et termini in singulis seriebus ultimos praecedentes notentur per $'X$, $'P$, $'Q$, $'R$ etc., habebimus pro y quasi alterum limitem

$$\begin{aligned} y=b+{}&\alpha\,(A+A'+A''+\cdots+{}'X)\\ &+\frac{1}{2}\alpha^2(B+B'+B''+\cdots+{}'P)\\ &+\frac{1}{6}\alpha^3(C+C'+C''+\cdots+{}'Q)\\ &+\frac{1}{24}\alpha^4(D+D'+D''+\cdots+{}'R)\\ &\text{etc.,}\end{aligned}$$

ita ut etiam in hac methodo emendata binos habebimus limites, inter quos verus valor ipsius y contineatur. Propius ergo valorem assequemur, si inter hos limites medium arithmeticum capiamus, unde prodibit

$$\begin{aligned} y=b+ &\quad \alpha(A+A'+A''+\cdots+X)-\frac{1}{2}\alpha(A+X)+\frac{1}{4}\alpha^2(B-P)\\ &+\frac{1}{6}\alpha^3(C+C'+C''+\cdots+Q)-\frac{1}{12}\alpha^3(C+Q)+\frac{1}{48}\alpha^4(D-R)\\ &+\frac{1}{120}\alpha^5(E+E'+E''+\cdots+S)-\frac{1}{240}\alpha^5(E+S)+\frac{1}{1440}\alpha^6(F-T)\\ &\text{etc.}\end{aligned}$$

Atque hinc superiores approximationes tantum addendo membrum $\frac{1}{4}\alpha^2(B-P)$ non mediocriter corrigentur.

EXEMPLUM 1

323. *Logarithmum cuiusvis numeri x proxime exprimere.*

Hic igitur est $y=\int\frac{dx}{x}$, quod integrale ita capitur, ut evanescat posito $x=1$; erit ergo $a=1$ et $b=0$ et $X=\frac{1}{x}$. Sumamus iam ab unitate ad x per intervalla $=\alpha$ ascendi, et cum sit

$$P=\frac{dX}{dx}=-\frac{1}{xx},\quad Q=\frac{dP}{dx}=\frac{2}{x^3},\quad R=\frac{dQ}{dx}=-\frac{6}{x^4},$$

pro indicibus

erit

$$\begin{array}{llllll} x= & 1, & 1+\alpha, & 1+2\alpha, & 1+3\alpha, & \cdots\ x\\ X= & 1, & \frac{1}{1+\alpha}, & \frac{1}{1+2\alpha}, & \frac{1}{1+3\alpha}, & \cdots\ \frac{1}{x},\\ P= & -1, & -\frac{1}{(1+\alpha)^2}, & -\frac{1}{(1+2\alpha)^2}, & -\frac{1}{(1+3\alpha)^2}, & \cdots -\frac{1}{xx},\\ Q= & 2, & \frac{2}{(1+\alpha)^3}, & \frac{2}{(1+2\alpha)^3}, & \frac{2}{(1+3\alpha)^3}, & \cdots\ \frac{2}{x^3},\\ R= & -6, & -\frac{6}{(1+\alpha)^4}, & -\frac{6}{(1+2\alpha)^4}, & -\frac{6}{(1+3\alpha)^4}, & \cdots -\frac{6}{x^4} \end{array}$$

etc.,

unde adipiscimur

$$\begin{aligned} lx = {} & \alpha\left(1+\frac{1}{1+\alpha}+\frac{1}{1+2\alpha}+\frac{1}{1+3\alpha}+\cdots+\frac{1}{x}\right) \\ & -\frac{1}{2}\alpha\left(1+\frac{1}{x}\right)-\frac{1}{4}\alpha\alpha\left(1-\frac{1}{xx}\right) \\ & +\frac{1}{3}\alpha^3\left(1+\frac{1}{(1+\alpha)^3}+\frac{1}{(1+2\alpha)^3}+\frac{1}{(1+3\alpha)^3}+\cdots+\frac{1}{x^3}\right) \\ & -\frac{1}{6}\alpha^3\left(1+\frac{1}{x^3}\right)-\frac{1}{8}\alpha^4\left(1-\frac{1}{x^4}\right) \\ & +\frac{1}{5}\alpha^5\left(1+\frac{1}{(1+\alpha)^5}+\frac{1}{(1+2\alpha)^5}+\frac{1}{(1+3\alpha)^5}+\cdots+\frac{1}{x^5}\right) \\ & -\frac{1}{10}\alpha^5\left(1+\frac{1}{x^5}\right)-\frac{1}{12}\alpha^6\left(1-\frac{1}{x^6}\right) \\ & \text{etc.} \end{aligned}$$

Quare si sumamus $\alpha = \frac{1}{m}$, erit

$$\begin{aligned} lx = {} & \left(\frac{1}{m}+\frac{1}{m+1}+\frac{1}{m+2}+\cdots+\frac{1}{mx}\right)-\frac{x+1}{2mx}-\frac{xx-1}{4mmxx} \\ & +\frac{1}{3}\left(\frac{1}{m^3}+\frac{1}{(m+1)^3}+\frac{1}{(m+2)^3}+\cdots+\frac{1}{(mx)^3}\right)-\frac{x^3+1}{6m^3x^3}-\frac{x^4-1}{8m^4x^4} \\ & +\frac{1}{5}\left(\frac{1}{m^5}+\frac{1}{(m+1)^5}+\frac{1}{(m+2)^5}+\cdots+\frac{1}{(mx)^5}\right)-\frac{x^5+1}{10m^5x^5}-\frac{x^6-1}{12m^6x^6} \\ & \text{etc.} \end{aligned}$$

COROLLARIUM

324. Si hae progressiones in infinitum continuentur, erit postremarum partium summa $= -\frac{1}{2}l\frac{m}{m-1}-\frac{1}{2}l\frac{mx+1}{mx} = -\frac{1}{2}l\frac{mx+1}{(m-1)x}$, primarum vero $= \frac{1}{2}l\frac{m+1}{m-1}$; unde, cum sit

$$lx+\frac{1}{2}l\frac{mx+1}{(m-1)x}+\frac{1}{2}l\frac{m-1}{m+1} = \frac{1}{2}l\frac{x(mx+1)}{m+1},$$

erit
$$l\,\frac{x(mx+1)}{m+1} = 2\left(\frac{1}{m+1}+\frac{1}{m+2}+\frac{1}{m+3}+\cdots+\frac{1}{mx}\right)$$
$$+\frac{2}{3}\left(\frac{1}{(m+1)^3}+\frac{1}{(m+2)^3}+\frac{1}{(m+3)^3}+\cdots+\frac{1}{m^3x^3}\right)$$
$$+\frac{2}{5}\left(\frac{1}{(m+1)^5}+\frac{1}{(m+2)^5}+\frac{1}{(m+3)^5}+\cdots+\frac{1}{m^5x^5}\right)$$
$$\text{etc.},$$
quae expressio adeo, si in infinitum continuetur, verum valorem log. $\frac{x(mx+1)}{m+1}$ praebet.

EXEMPLUM 2

325. *Arcum circuli, cuius tangens est* $=\frac{x}{c}$, *hac methodo proxime exprimere.*

Quaestio igitur est de integrali $y=\int\frac{c\,dx}{cc+xx}$, quod posito $x=0$ evanescit, eritque $a=0$ et $b=0$, tum vero
$$X=\frac{c}{cc+xx},\quad P=\frac{dX}{dx}=\frac{-2cx}{(cc+xx)^2},\quad Q=\frac{dP}{dx}=\frac{-2c(cc-3xx)}{(cc+xx)^3},$$
$$R=\frac{dQ}{dx}=\frac{6cx(3cc-4xx)}{(cc+xx)^4},\quad S=\frac{dR}{dx}=\frac{6c(3c^4-33ccxx+20x^4)}{(cc+xx)^5}\ \text{etc.},$$
quae formae in infinitum continuatae dant
$$y=\frac{cx}{cc+xx}+\frac{cx^3}{(cc+xx)^2}-\frac{cx^3(cc-3xx)}{3(cc+xx)^3}-\frac{cx^5(3cc-4xx)}{4(cc+xx)^4}$$
$$+\frac{cx^5(3c^4-33ccxx+20x^4)}{20(cc+xx)^5}+\text{etc.}$$
Verum si x per intervalla $=1$, ut sit $\alpha=1$, crescere ponamus, erit
$$A=\frac{c}{cc},\quad B=0\quad,\quad C=\frac{-2c^3}{c^6}\quad,\quad D=0\quad\text{etc.}$$
$$A'=\frac{c}{cc+1},\quad B'=\frac{-2c}{(cc+1)^2},\quad C'=\frac{-2c(cc-3)}{(cc+1)^3},\quad D'=\frac{6c(3cc-4)}{(cc+1)^4},$$
$$A''=\frac{c}{cc+4},\quad B''=\frac{-4c}{(cc+4)^2},\quad C''=\frac{-2c(cc-12)}{(cc+4)^3},\quad D''=\frac{12c(3cc-16)}{(cc+4)^4},$$
$$A'''=\frac{c}{cc+9},\quad B'''=\frac{-6c}{(cc+9)^2},\quad C'''=\frac{-2c(cc-27)}{(cc+9)^3},\quad D'''=\frac{18c(3cc-36)}{(cc+9)^4},$$
$$\vdots$$
$$X=\frac{c}{cc+xx},\quad P=\frac{-2cx}{(cc+xx)^2},\quad Q=\frac{-2c(cc-3xx)}{(cc+xx)^3},\quad R=\frac{6cx(3cc-4xx)}{(cc+xx)^4}$$

hincque

$$y = c\left(\frac{1}{cc} + \frac{1}{cc+1} + \frac{1}{cc+4} + \frac{1}{cc+9} + \cdots + \frac{1}{cc+xx}\right)$$
$$- \frac{1}{2c} - \frac{c}{2(cc+xx)} + \frac{cx}{2(cc+xx)^2}$$
$$- \frac{c}{3}\left(\frac{1}{c^4} + \frac{cc-3}{(cc+1)^3} + \frac{cc-12}{(cc+4)^3} + \frac{cc-27}{(cc+9)^3} + \cdots + \frac{cc-3xx}{(cc+xx)^3}\right)$$
$$+ \frac{1}{6c^3} + \frac{c(cc-3xx)}{6(cc+xx)^3} - \frac{cx(3cc-4xx)}{8(cc+xx)^4}$$

etc.

COROLLARIUM

326. Posito ergo $c = x = 4$, ut fiat $y = \text{Ang. tang.}\, 1 = \frac{\pi}{4}$, erit

$$\frac{\pi}{4} = \frac{1}{4} + \frac{4}{17} + \frac{4}{20} + \frac{4}{25} + \frac{1}{8} - \frac{1}{8} - \frac{1}{16} + \frac{1}{128}$$
$$- \frac{4}{3}\left(\frac{1}{256} + \frac{13}{17^3} + \frac{4}{20^3} - \frac{11}{25^3} - \frac{32}{32^3}\right) + \frac{1}{384} - \frac{1}{1536} + \frac{1}{128 \cdot 256},$$

cuius valor non multum a veritate discedit; sed haec exempla tantum illustrationis causa affero, non ut approximatio facilior, quam aliae methodi suppeditant, inde expectetur.

EXEMPLUM 3

327. *Integrale* $y = \int \frac{e^{-\frac{1}{x}}dx}{x}$ *ita sumtum, ut evanescat posito* $x = 0$, *vero proxime assignare.*

Per reductiones supra expositas est

$$\int \frac{e^{-\frac{1}{x}}dx}{x} = e^{-\frac{1}{x}}x - \int e^{-\frac{1}{x}}dx$$

et pars $e^{-\frac{1}{x}}x$ evanescit posito $x = 0$. Quaeramus ergo integrale $z = \int e^{-\frac{1}{x}}dx$, quia eo invento habetur $y = e^{-\frac{1}{x}}x - z$, ac supra iam observavimus alias methodos approximandi in hoc exemplo frustra tentari. Cum igitur posito

$x = 0$ evanescat z, erit $a = 0$ et $b = 0$, tum vero $X = e^{-\frac{1}{x}}$ hincque

$$P = \frac{dX}{dx} = e^{-\frac{1}{x}} \frac{1}{xx}, \quad Q = \frac{dP}{dx} = e^{-\frac{1}{x}}\left(\frac{1}{x^4} - \frac{2}{x^3}\right), \quad R = \frac{dQ}{dx} = e^{-\frac{1}{x}}\left(\frac{1}{x^6} - \frac{6}{x^5} + \frac{6}{x^4}\right),$$

$$S = \frac{dR}{dx} = e^{-\frac{1}{x}}\left(\frac{1}{x^8} - \frac{12}{x^7} + \frac{36}{x^6} - \frac{24}{x^5}\right) \quad \text{etc.},$$

quibus valoribus in infinitum continuatis erit

$$z = e^{-\frac{1}{x}} \left\{ \begin{aligned} &x - \frac{1}{2} + \frac{1}{6} x^3 \left(\frac{1}{x^4} - \frac{2}{x^3}\right) - \frac{1}{24} x^4 \left(\frac{1}{x^6} - \frac{6}{x^5} + \frac{6}{x^4}\right) \\ &+ \frac{1}{120} x^5 \left(\frac{1}{x^8} - \frac{12}{x^7} + \frac{36}{x^6} - \frac{24}{x^5}\right) - \text{etc.} \end{aligned} \right\}$$

seu

$$z = e^{-\frac{1}{x}} \left\{ \begin{aligned} &x - \frac{1}{2} + \frac{1}{6}\left(\frac{1}{x} - 2\right) - \frac{1}{24}\left(\frac{1}{xx} - \frac{6}{x} + 6\right) + \frac{1}{120}\left(\frac{1}{x^3} - \frac{12}{x^2} + \frac{36}{x} - 24\right) \\ &- \frac{1}{720}\left(\frac{1}{x^4} - \frac{20}{x^3} + \frac{120}{x^2} - \frac{240}{x} + 120\right) + \text{etc.} \end{aligned} \right\}$$

quae series parum convergit, quicunque valor ipsi x tribuatur. Per intervalla igitur a 0 usque ad x ascendamus ponendo pro x successive 0, α, 2α, 3α etc., ubi notandum fore $A = 0$, $B = 0$, $C = 0$, $D = 0$ etc., ac regula nostra praebet

$$z = \alpha\left(e^{-\frac{1}{\alpha}} + e^{-\frac{1}{2\alpha}} + e^{-\frac{1}{3\alpha}} + \cdots + e^{-\frac{1}{x}}\right) - \frac{1}{2}\alpha e^{-\frac{1}{x}} - \frac{1}{4}\alpha^2 e^{-\frac{1}{x}} \frac{1}{xx}$$

$$+ \frac{1}{6}\alpha^3\left(e^{-\frac{1}{\alpha}}\left(\frac{1}{\alpha^4} - \frac{2}{\alpha^3}\right) + e^{-\frac{1}{2\alpha}}\left(\frac{1}{16\alpha^4} - \frac{2}{8\alpha^3}\right) + e^{-\frac{1}{3\alpha}}\left(\frac{1}{81\alpha^4} - \frac{2}{27\alpha^3}\right) + \cdots + e^{-\frac{1}{x}}\left(\frac{1}{x^4} - \frac{2}{x^3}\right)\right)$$

$$- \frac{1}{12}\alpha^3 e^{-\frac{1}{x}}\left(\frac{1}{x^4} - \frac{2}{x^3}\right) - \frac{1}{48}\alpha^4 e^{-\frac{1}{x}}\left(\frac{1}{x^6} - \frac{6}{x^5} + \frac{6}{x^4}\right).$$

Si hinc valorem ipsius z pro casu $x = 1$ determinare velimus et pro α fractionem parvam $\frac{1}{n}$ assumamus, habebimus

$$z = \frac{1}{n}\left(e^{-\frac{n}{1}} + e^{-\frac{n}{2}} + e^{-\frac{n}{3}} + e^{-\frac{n}{4}} + \cdots + e^{-\frac{n}{n}}\right) - \frac{1}{2ne} - \frac{1}{4nne}$$

$$+ \frac{1}{6}\left(e^{-\frac{n}{1}}\frac{n-2}{1} + e^{-\frac{n}{2}}\frac{n-4}{16} + e^{-\frac{n}{3}}\frac{n-6}{81} + \cdots + e^{-\frac{n}{n}}\frac{n-2n}{n^4}\right)$$

$$+ \frac{1}{12n^3e} - \frac{1}{48n^4e}.$$

Si hic pro n sumatur numerus mediocriter magnus, veluti 10, valor ipsius z ad partem millionesimam unitatis exactus reperitur ac vicies exactior prodiret, si pro n sumeremus 20.

SCHOLION 1

328. Hoc exemplum sufficiat eximium usum huius methodi approximandi ostendisse. Interim tamen occurrunt casus, quibus ne hac quidem methodo uti licet, etiamsi totum spatium, per quod variabilis x crescit, in minima intervalla dividamus. Evenit hoc, quando functio X pro quopiam intervallo, dum variabili x certus quidam valor tribuitur, in infinitum excrescit, cum tamen ipsa quantitas integralis $y = \int X dx$ hoc casu non fiat infinita; veluti si fuerit

$$y = \int \frac{dx}{\sqrt{(a-x)}},$$

ubi $X = \frac{1}{\sqrt{(a-x)}}$, quae posito $x = a$ fit infinita, integrale vero $y = C - 2\sqrt{(a-x)}$ hoc casu est finitum. Hoc autem semper usu venit, quoties huiusmodi factor $a - x$ in denominatore habet exponentem unitate minorem; tum enim idem factor in integrali in numeratorem transit; sin autem eiusdem factoris exponens in denominatore est unitas vel adeo unitate maior, tum etiam ipsum integrale casu $x = a$ fit infinitum; quo casu quia approximatio cessat, hic tantum de iis sermo est, ubi exponens unitate est minor, quoniam tum approximatio revera turbatur. Verum huic incommodo facile medela afferri potest; cum enim differentiale eiusmodi formam sit habiturum $\frac{X dx}{(a-x)^{\lambda:\mu}}$ existente $\lambda < \mu$, ponatur $a - x = z^\mu$, ut sit $x = a - z^\mu$ et $dx = -\mu z^{\mu-1} dz$, et differentiale nostrum erit $= -\mu X z^{\mu-\lambda-1} dz$, quod casu $x = a$ seu $z = 0$ non amplius fit infinitum. Vel, quod eodem redit, pro iis intervallis, quibus functio X fit infinita, integratio seorsim revera instituatur ponendo $x = a \pm \omega$; tum enim formula $X dx$ satis fiet simplex ob ω valde parvum, ut integratio nihil habeat difficultatis. Veluti si valorem ipsius $y = \int \frac{xx\,dx}{\sqrt{(a^4 - x^4)}}$ per intervalla ab $x = 0$ usque ad $x = a - \alpha$ iam simus consecuti, pro hoc ultimo intervallo ponamus $x = a - \omega$ et integrari oportebit

$$\frac{(a-\omega)^2 d\omega}{\sqrt{(4a^3\omega - 6aa\omega\omega + 4a\omega^3 - \omega^4)}},$$

quod ob ω valde parvum abit in

$$\frac{d\omega\sqrt{a}}{2\sqrt{\omega}}\left(1 - \frac{5\omega}{4a} - \frac{5\omega\omega}{32aa}\right),$$

cuius integrale sumto $\omega = \alpha$ est

$$\sqrt{a\alpha} - \frac{5\alpha\sqrt{\alpha}}{12\sqrt{a}} - \frac{\alpha\alpha\sqrt{\alpha}}{32a\sqrt{a}}\ ^{1)},$$

quod, si ad plures terminos continuetur, non solum pro ultimo intervallo, sed pro duobus pluribusve postremis ponendo $\omega = 2\alpha$ vel $\omega = 3\alpha$ adhiberi potest. Pro quibus enim intervallis denominator iam fit satis parvus, praestat hac methodo uti quam ea, quae ante est exposita.

SCHOLION 2

329. Interdum etiam aliud incommodum occurrit, ut denominator duobus casibus evanescat, veluti si fuerit

$$y = \int \frac{X\,dx}{\sqrt{(a-x)(x-b)}},$$

ubi variabilis x semper inter limites b et a contineri debet, ita ut, cum a b ad a creverit, deinceps iterum ab a ad b decrescat; interea autem integrale y continuo crescere pergat, cuius igitur valor per intervalla commode determinari non potest. Hoc ergo casu in subsidium vocetur haec substitutio

$$x = \frac{1}{2}(a+b) - \frac{1}{2}(a-b)\cos.\varphi,$$

1) Editio princeps: *quod ob* ω *valde parvum abit in* $\frac{d\omega\sqrt{a}}{2\sqrt{\omega}}\left(1 - \frac{\omega}{2a} + \frac{7\omega\omega}{8aa}\right)$ *cuius integrale sumto* $\omega = \alpha$ *est* $\sqrt{a\alpha} - \frac{\alpha\sqrt{\alpha}}{6\sqrt{a}} + \frac{7\alpha\alpha\sqrt{\alpha}}{4aa\sqrt{a}}$. Correxit L. S.

qua fit

$$dx = \frac{1}{2}(a-b)d\varphi \sin.\varphi$$

et

$$(a-x)(x-b) = \left(\tfrac{1}{2}(a-b) + \tfrac{1}{2}(a-b)\cos.\varphi\right)\left(\tfrac{1}{2}(a-b) - \tfrac{1}{2}(a-b)\cos.\varphi\right)$$

seu

$$(a-x)(x-b) = \tfrac{1}{4}(a-b)^2 \sin.\varphi^2,$$

unde oritur $y = \int X d\varphi$, quae nullo amplius incommodo laborat, cum angulum φ continuo ulterius aequabiliter augere licet.

Hoc etiam ad casus patet, ubi bini factores in denominatore non eundem habent exponentem, veluti si fuerit

$$y = \int \frac{X dx}{\sqrt[2\lambda]{(a-x)^\mu (x-b)^\nu}},$$

ita ut μ et ν sint minores quam 2λ, quem exponentem parem suppono. Si iam μ et ν non sint aequales, sed $\nu < \mu$, ad aequalitatem reducantur hoc modo

$$y = \int \frac{X dx \sqrt[2\lambda]{(x-b)^{\mu-\nu}}}{\sqrt[2\lambda]{(a-x)^\mu (x-b)^\mu}}.$$

Quodsi iam ut ante ponatur

$$x = \tfrac{1}{2}(a+b) - \tfrac{1}{2}(a-b)\cos.\varphi,$$

obtinebitur

$$y = \left(\frac{a-b}{2}\right)^{\frac{2\lambda-\mu-\nu}{2\lambda}} \int X d\varphi \sin.\varphi^{\frac{\lambda-\mu}{\lambda}} (1-\cos.\varphi)^{\frac{\mu-\nu}{2\lambda}},$$

ubi angulum φ, quousque libuerit, continuare et methodo per intervalla procedente uti licet. Quibus observatis vix quicquam amplius hanc methodum approximandi remorabitur.

CAPUT VIII

DE VALORIBUS INTEGRALIUM QUOS CERTIS TANTUM CASIBUS RECIPIUNT

PROBLEMA 38

330. *Integralis* $\int \frac{x^m dx}{\sqrt{(1-xx)}}$ *valorem, quem posito* $x=1$ *recipit, assignare, integrali scilicet ita determinato, ut evanescat posito* $x=0$.

SOLUTIO

Pro casibus simplicissimis, quibus $m=0$ vel $m=1$, habemus posito $x=1$ post integrationem

$$\int \frac{dx}{\sqrt{(1-xx)}} = \frac{\pi}{2} \quad \text{et} \quad \int \frac{x\,dx}{\sqrt{(1-xx)}} = 1.$$

Deinde supra § 120 vidimus esse in genere

$$\int \frac{x^{m+1}dx}{\sqrt{(1-xx)}} = \frac{m}{m+1}.\int \frac{x^{m-1}dx}{\sqrt{(1-xx)}} - \frac{1}{m+1}x^m\sqrt{(1-xx)};$$

casu ergo $x=1$ erit

$$\int \frac{x^{m+1}dx}{\sqrt{(1-xx)}} = \frac{m}{m+1}\int \frac{x^{m-1}dx}{\sqrt{(1-xx)}}$$

unde a simplicissimis ad maiores exponentis m valores progrediendo obtinebimus

$$\int\frac{dx}{\sqrt{(1-xx)}}=\frac{\pi}{2}$$

$$\int\frac{xx\,dx}{\sqrt{(1-xx)}}=\frac{1}{2}\cdot\frac{\pi}{2}$$

$$\int\frac{x^4dx}{\sqrt{(1-xx)}}=\frac{1\cdot 3}{2\cdot 4}\cdot\frac{\pi}{2}$$

$$\int\frac{x^6dx}{\sqrt{(1-xx)}}=\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{\pi}{2}$$

$$\int\frac{x^8dx}{\sqrt{(1-xx)}}=\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{\pi}{2}$$

$$\vdots$$

$$\int\frac{x^{2n}dx}{\sqrt{(1-xx)}}=\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots\cdots 2n}\cdot\frac{\pi}{2}$$

$$\int\frac{x\,dx}{\sqrt{(1-xx)}}=1$$

$$\int\frac{x^3dx}{\sqrt{(1-xx)}}=\frac{2}{3}$$

$$\int\frac{x^5dx}{\sqrt{(1-xx)}}=\frac{2\cdot 4}{3\cdot 5}$$

$$\int\frac{x^7dx}{\sqrt{(1-xx)}}=\frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}$$

$$\int\frac{x^9dx}{\sqrt{(1-xx)}}=\frac{2\cdot 4\cdot 6\cdot 8}{3\cdot 5\cdot 7\cdot 9}$$

$$\vdots$$

$$\int\frac{x^{2n+1}dx}{\sqrt{(1-xx)}}=\frac{2\cdot 4\cdot 6\cdots\cdots 2n}{3\cdot 5\cdot 7\cdots(2n+1)}$$

COROLLARIUM 1

331. Integrale ergo $\int\frac{x^m dx}{\sqrt{(1-xx)}}$ posito $x=1$ algebraice exprimitur casibus, quibus exponens m est numerus integer impar, casibus autem, quibus est par, quadraturam circuli involvit; semper enim π designat peripheriam circuli, cuius diameter $=1$.

COROLLARIUM 2

332. Si binas postremas formulas in se multiplicemus, prodit

$$\int\frac{x^{2n}dx}{\sqrt{(1-xx)}}\cdot\int\frac{x^{2n+1}dx}{\sqrt{(1-xx)}}=\frac{1}{2n+1}\cdot\frac{\pi}{2},$$

posito scilicet $x=1$, quam veram esse patet, etiamsi n non sit numerus integer.[1])

1) Si more consueto ponitur $\int_0^1 x^{p-1}(1-x)^{q-1}dx=B(p,q)$, $\int_0^\infty e^{-x}x^{p-1}dx=\Gamma(p)$, erit pro quocunque valore ipsius n posito $x=1$

$$\int\frac{x^{2n}dx}{\sqrt{(1-xx)}}\cdot\int\frac{x^{2n+1}dx}{\sqrt{(1-xx)}}=\frac{1}{4}B\left(n+\frac{1}{2},\ \frac{1}{2}\right)B\left(n+1,\ \frac{1}{2}\right)=\frac{1}{4}\,\frac{\Gamma(n+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(n+1)}\cdot\frac{\Gamma(n+1)\cdot\Gamma(\frac{1}{2})}{\Gamma(n+\frac{3}{2})}$$
$$=\frac{\pi}{2}\cdot\frac{1}{2n+1}.$$

Vide L. EULERI Commentationes 321, 421, 640 et 675 (indicis ENESTROEMIANI), *LEONHARDI EULERI Opera omnia*, series I, vol. 17, 18 et 19. L. S.

COROLLARIUM 3

333. Haec ergo aequalitas subsistet, si ponamus $x = z^\nu$, iisdem conditionibus, quia sumto $x = 0$ vel $x = 1$ fit $z = 0$ vel $z = 1$. Erit ergo

$$\nu\nu \int \frac{z^{2n\nu+\nu-1}dz}{\sqrt{(1-z^{2\nu})}} \cdot \int \frac{z^{2n\nu+2\nu-1}dz}{\sqrt{(1-z^{2\nu})}} = \frac{1}{2n+1} \cdot \frac{\pi}{2}$$

et posito $2n\nu + \nu - 1 = \mu$ fiet posito $z = 1$

$$\int \frac{z^\mu dz}{\sqrt{(1-z^{2\nu})}} \cdot \int \frac{z^{\mu+\nu} dz}{\sqrt{(1-z^{2\nu})}} = \frac{1}{\nu(\mu+1)} \cdot \frac{\pi}{2}.$$

SCHOLION 1

334. Quod tale productum binorum integralium exhiberi queat, eo magis est notatu dignum, quod aequalitas haec subsistit, etiamsi neutra formula neque algebraice neque per π exhiberi queat. Veluti si $\nu = 2$ et $\mu = 0$, fit

$$\int \frac{dz}{\sqrt{(1-z^4)}} \cdot \int \frac{zz\,dz}{\sqrt{(1-z^4)}} = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}$$

similique modo, [si]

$\nu = 3, \quad \mu = 0, \quad$ fit $\displaystyle\int \frac{dz}{\sqrt{(1-z^6)}} \cdot \int \frac{z^3 dz}{\sqrt{(1-z^6)}} = \frac{1}{3} \cdot \frac{\pi}{2} = \frac{\pi}{6},$

$\nu = 3, \quad \mu = 1, \quad$ fit $\displaystyle\int \frac{z\,dz}{\sqrt{(1-z^6)}} \cdot \int \frac{z^4 dz}{\sqrt{(1-z^6)}} = \frac{1}{6} \cdot \frac{\pi}{2} = \frac{\pi}{12},$

$\nu = 4, \quad \mu = 0, \quad$ fit $\displaystyle\int \frac{dz}{\sqrt{(1-z^8)}} \cdot \int \frac{z^4 dz}{\sqrt{(1-z^8)}} = \frac{1}{4} \cdot \frac{\pi}{2} = \frac{\pi}{8},$

$\nu = 4, \quad \mu = 2, \quad$ fit $\displaystyle\int \frac{zz\,dz}{\sqrt{(1-z^8)}} \cdot \int \frac{z^6 dz}{\sqrt{(1-z^8)}} = \frac{1}{12} \cdot \frac{\pi}{2} = \frac{\pi}{24},$

$\nu = 5, \quad \mu = 0, \quad$ fit $\displaystyle\int \frac{dz}{\sqrt{(1-z^{10})}} \cdot \int \frac{z^5 dz}{\sqrt{(1-z^{10})}} = \frac{1}{5} \cdot \frac{\pi}{2} = \frac{\pi}{10},$

$\nu = 5, \quad \mu = 1, \quad$ fit $\displaystyle\int \frac{z\,dz}{\sqrt{(1-z^{10})}} \cdot \int \frac{z^6 dz}{\sqrt{(1-z^{10})}} = \frac{1}{10} \cdot \frac{\pi}{2} = \frac{\pi}{20},$

$\nu = 5, \quad \mu = 2, \quad$ fit $\displaystyle\int \frac{zz\,dz}{\sqrt{(1-z^{10})}} \cdot \int \frac{z^7 dz}{\sqrt{(1-z^{10})}} = \frac{1}{15} \cdot \frac{\pi}{2} = \frac{\pi}{30},$

$\nu = 5, \quad \mu = 3, \quad$ fit $\displaystyle\int \frac{z^3 dz}{\sqrt{(1-z^{10})}} \cdot \int \frac{z^8 dz}{\sqrt{(1-z^{10})}} = \frac{1}{20} \cdot \frac{\pi}{2} = \frac{\pi}{40},$

quae theoremata sine dubio omni attentione sunt digna.

SCHOLION 2

335. Facile hinc etiam colligitur valor integralis $\int \frac{x^m dx}{\sqrt{(x-xx)}}$ posito $x=1$; si enim scribamus $x=zz$, fiet hoc integrale $2\int \frac{z^{2m} dz}{\sqrt{(1-zz)}}$, quocirca pro casu $x=1$ nanciscimur sequentes valores

$$\int \frac{dx}{\sqrt{(x-xx)}} = \pi, \qquad \int \frac{x^4 dx}{\sqrt{(x-xx)}} = \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\cdot \pi,$$

$$\int \frac{x dx}{\sqrt{(x-xx)}} = \frac{1}{2}\cdot \pi, \qquad \int \frac{x^5 dx}{\sqrt{(x-xx)}} = \frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8\cdot 10}\cdot \pi,$$

$$\int \frac{xx dx}{\sqrt{(x-xx)}} = \frac{1\cdot 3}{2\cdot 4}\cdot \pi, \qquad \vdots$$

$$\int \frac{x^3 dx}{\sqrt{(x-xx)}} = \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot \pi, \qquad \int \frac{x^m dx}{\sqrt{(x-xx)}} = \frac{1\cdot 3\cdot 5\cdots (2m-1)}{2\cdot 4\cdot 6\cdots\cdots 2m}\cdot \pi.$$

Hinc ergo integralium huiusmodi formulas involventium, quae magis sunt complicata, valores, quos posito $x=1$ recipiunt, per series succincte exprimi possunt, quem usum aliquot exemplis declaremus.

EXEMPLUM 1

336. *Valorem integralis* $\int \frac{dx}{\sqrt{(1-x^4)}}$ *posito* $x=1$ *per seriem exhibere.*

Integrali detur haec forma[1])

$$\int \frac{dx}{\sqrt{(1-xx)}}\cdot (1+xx)^{-\frac{1}{2}},$$

ut habeamus

$$\int \frac{dx}{\sqrt{(1-x^4)}} = \int \frac{dx}{\sqrt{(1-xx)}}\left(1-\frac{1}{2}xx+\frac{1\cdot 3}{2\cdot 4}x^4-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}x^8-\text{etc.}\right);$$

singulis ergo terminis pro casu $x=1$ integratis orietur

$$\int \frac{dx}{\sqrt{(1-x^4)}} = \frac{\pi}{2}\left(1-\frac{1}{4}+\frac{1\cdot 9}{4\cdot 16}-\frac{1\cdot 9\cdot 25}{4\cdot 16\cdot 36}+\frac{1\cdot 9\cdot 25\cdot 49}{4\cdot 16\cdot 36\cdot 64}-\text{etc.}\right).$$

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1) Vide L. EULERI Commentationem 605 (indicis ENESTROEMIANI): *De miris proprietatibus curvae elasticae sub aequatione* $y=\int \frac{xx dx}{\sqrt{(1-x^4)}}$ *contentae*, Acta acad. sc. Petrop. 1782: II (1786), p. 34; *LEONHARDI EULERI Opera omnia*, series I, vol. 21, p. 91, imprimis p. 97. L. S.

COROLLARIUM

337. Simili modo pro eodem casu $x = 1$ reperitur

$$\int \frac{x\,dx}{\sqrt{(1-x^4)}} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \text{etc.} = \frac{\pi}{4},$$

$$\int \frac{xx\,dx}{\sqrt{(1-x^4)}} = \frac{\pi}{2}\left(\frac{1}{2} - \frac{1^2\cdot 3}{2^2\cdot 4} + \frac{1^2\cdot 3^2\cdot 5}{2^2\cdot 4^2\cdot 6} - \frac{1^2\cdot 3^2\cdot 5^2\cdot 7}{2^2\cdot 4^2\cdot 6^2\cdot 8} + \text{etc.}\right),$$

$$\int \frac{x^3\,dx}{\sqrt{(1-x^4)}} = \frac{2}{3} - \frac{4}{3\cdot 5} + \frac{6}{5\cdot 7} - \frac{8}{7\cdot 9} + \frac{10}{9\cdot 11} - \text{etc.};$$

est autem

$$\int \frac{x^3\,dx}{\sqrt{(1-x^4)}} = \frac{1}{2} - \frac{1}{2}\sqrt{(1-x^4)}$$

ideoque $= \frac{1}{2}$ posito $x = 1$, unde haec postrema series est $= \frac{1}{2}$.

EXEMPLUM 2

338. *Valorem integralis* $\int dx \sqrt{\frac{1+axx}{1-xx}}$ *casu* $x = 1$ *per seriem exhibere.*

Cum sit

$$\sqrt{(1+a\,xx)} = 1 + \frac{1}{2}axx - \frac{1\cdot 1}{2\cdot 4}a^2x^4 + \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}a^3x^6 - \text{etc.},$$

erit per $\int \frac{dx}{\sqrt{(1-xx)}}$ multiplicando et integrando

$$\int dx \sqrt{\frac{1+axx}{1-xx}} = \frac{\pi}{2}\left(1 + \frac{1\cdot 1}{2\cdot 2}a - \frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4}a^2 + \frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}a^3 - \text{etc.}\right),$$

unde peripheriam ellipsis cognoscere licet.[1])

EXEMPLUM 3

339. *Valorem integralis* $\int \frac{dx}{\sqrt{x(1-xx)}}$ *casu* $x = 1$ *per seriem exhibere.*

Repraesentetur haec formula ita $\int \frac{dx(1+x)^{-\frac{1}{2}}}{\sqrt{(x-xx)}}$, ut sit

$$= \int \frac{dx}{\sqrt{(x-xx)}}\left(1 - \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^3 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \text{etc.}\right),$$

1) Vide L. EULERI Commentationem 154 (indicis ENESTROEMIANI): *Animadversiones in rectificationem ellipsis*, Opusc. var. arg. 2, 1750, p. 121; *LEONHARDI EULERI Opera omnia*, series I, vol. 20, p. 21, imprimis p. 25. L. S.

unde series haec obtinetur

$$\int \frac{dx}{\sqrt{x(1-xx)}} = \pi\left(1 - \frac{1}{4} + \frac{1\cdot 9}{4\cdot 16} - \frac{1\cdot 9\cdot 25}{4\cdot 16\cdot 36} + \text{etc.}\right),$$

quae ab exemplo primo haud differt; quod non mirum, cum posito $x = zz$ haec formula ad illam reducatur.

PROBLEMA 39

340. *Valorem integralis* $\int x^{m-1}dx(1-xx)^{n-\frac{1}{2}}$, *quod posito* $x = 0$ *evanescat,* [*casu* $x = 1$] *definire.*

SOLUTIO

Reductiones supra § 118 datae praebent pro hoc casu

$$\int x^{m-1}dx(1-xx)^{\frac{\mu}{2}+1} = \frac{x^m(1-xx)^{\frac{\mu}{2}+1}}{m+\mu+2} + \frac{\mu+2}{m+\mu+2}\int x^{m-1}dx(1-xx)^{\frac{\mu}{2}};$$

sumto ergo $\mu = 2n-1$ erit

$$\int x^{m-1}dx(1-xx)^{n+\frac{1}{2}} = \frac{2n+1}{m+2n+1}\int x^{m-1}dx(1-xx)^{n-\frac{1}{2}}$$

posito $x = 1$. Cum igitur in praecedente problemate valor $\int \frac{x^{m-1}dx}{\sqrt{(1-xx)}}$ sit assignatus, quam brevitatis gratia ponamus $= M$, hinc ad sequentes progrediamur

$$\int \frac{x^{m-1}dx}{\sqrt{(1-xx)}} = M,$$

$$\int x^{m-1}dx\,(1-xx)^{\frac{1}{2}} = \frac{1}{m+1}M,$$

$$\int x^{m-1}dx\,(1-xx)^{\frac{3}{2}} = \frac{1\cdot 3}{(m+1)(m+3)}M,$$

$$\int x^{m-1}dx(1-xx)^{\frac{5}{2}} = \frac{1\cdot 3\cdot 5}{(m+1)(m+3)(m+5)}M$$

et in genere

$$\int x^{m-1}dx(1-xx)^{n-\frac{1}{2}} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{(m+1)(m+3)(m+5)\cdots(m+2n-1)}M.$$

Iam duo casus sunt perpendendi, prout $m-1$ est vel numerus par vel impar; si enim $m-1$ sit par, erit

$$M=\frac{1\cdot 3\cdot 5\cdots(m-2)}{2\cdot 4\cdot 6\cdots(m-1)}\cdot\frac{\pi}{2};$$

[sin autem] $m-1$ sit impar, erit

$$M=\frac{2\cdot 4\cdot 6\cdots(m-2)}{3\cdot 5\cdot 7\cdots(m-1)}.$$

Hinc sequentes deducuntur valores

$\int dx\sqrt{(1-xx)}=\frac{\pi}{4}$	$\int x\,dx\sqrt{(1-xx)}=\frac{1}{3}$
$\int xxdx\sqrt{(1-xx)}=\frac{1}{4}\cdot\frac{\pi}{4}$	$\int x^3dx\sqrt{(1-xx)}=\frac{1}{3}\cdot\frac{2}{5}$
$\int x^4dx\sqrt{(1-xx)}=\frac{1\cdot 3}{4\cdot 6}\cdot\frac{\pi}{4}$	$\int x^5dx\sqrt{(1-xx)}=\frac{1}{3}\cdot\frac{2\cdot 4}{5\cdot 7}$
$\int x^6dx\sqrt{(1-xx)}=\frac{1\cdot 3\cdot 5}{4\cdot 6\cdot 8}\cdot\frac{\pi}{4}$	$\int x^7dx\sqrt{(1-xx)}=\frac{1}{3}\cdot\frac{2\cdot 4\cdot 6}{5\cdot 7\cdot 9}$
$\int dx(1-xx)^{\frac{3}{2}}=\frac{3\pi}{16}$	$\int xdx(1-xx)^{\frac{3}{2}}=\frac{1}{5}$
$\int xxdx(1-xx)^{\frac{3}{2}}=\frac{1}{6}\cdot\frac{3\pi}{16}$	$\int x^3dx(1-xx)^{\frac{3}{2}}=\frac{1}{5}\cdot\frac{2}{7}$
$\int x^4dx(1-xx)^{\frac{3}{2}}=\frac{1\cdot 3}{6\cdot 8}\cdot\frac{3\pi}{16}$	$\int x^5dx(1-xx)^{\frac{3}{2}}=\frac{1}{5}\cdot\frac{2\cdot 4}{7\cdot 9}$
$\int x^6dx(1-xx)^{\frac{3}{2}}=\frac{1\cdot 3\cdot 5}{6\cdot 8\cdot 10}\cdot\frac{3\pi}{16}$	$\int x^7dx(1-xx)^{\frac{3}{2}}=\frac{1}{5}\cdot\frac{2\cdot 4\cdot 6}{7\cdot 9\cdot 11}$
$\int dx(1-xx)^{\frac{5}{2}}=\frac{5\pi}{32}$	$\int xdx(1-xx)^{\frac{5}{2}}=\frac{1}{7}$
$\int x^2dx(1-xx)^{\frac{5}{2}}=\frac{1}{8}\cdot\frac{5\pi}{32}$	$\int x^3dx(1-xx)^{\frac{5}{2}}=\frac{1}{7}\cdot\frac{2}{9}$
$\int x^4dx(1-xx)^{\frac{5}{2}}=\frac{1\cdot 3}{8\cdot 10}\cdot\frac{5\pi}{32}$	$\int x^5dx(1-xx)^{\frac{5}{2}}=\frac{1}{7}\cdot\frac{2\cdot 4}{9\cdot 11}$
$\int x^6dx(1-xx)^{\frac{5}{2}}=\frac{1\cdot 3\cdot 5}{8\cdot 10\cdot 12}\cdot\frac{5\pi}{32}$	$\int x^7dx(1-xx)^{\frac{5}{2}}=\frac{1}{7}\cdot\frac{2\cdot 4\cdot 6}{9\cdot 11\cdot 13}$

etc.

PROBLEMA 40

341. *Valores integralium* $\int \frac{x^m dx}{\sqrt[3]{(1-x^3)}}$ *et* $\int \frac{x^m dx}{\sqrt[3]{(1-x^3)^2}}$ [*ita determinatorum, ut posito* $x=0$ *evanescant,*] *posito* $x=1$ *assignare.*

SOLUTIO

Ponamus pro casibus simplicissimis

$$\int \frac{dx}{\sqrt[3]{(1-x^3)}} = A, \quad \int \frac{x\,dx}{\sqrt[3]{(1-x^3)}} = B, \quad \int \frac{xx\,dx}{\sqrt[3]{(1-x^3)}} = C,$$

$$\int \frac{dx}{\sqrt[3]{(1-x^3)^2}} = A', \quad \int \frac{x\,dx}{\sqrt[3]{(1-x^3)^2}} = B', \quad \int \frac{xx\,dx}{\sqrt[3]{(1-x^3)^2}} = C'$$

et ex reductione prima § 118 posito $a=1$ et $b=-1$ pro casu $x=1$ habemus

$$\int x^{m+n-1}dx(1-x^n)^{\frac{\mu}{\nu}} = \frac{m\nu}{m\nu+n\mu+n\nu}\int x^{m-1}dx(1-x^n)^{\frac{\mu}{\nu}};$$

ergo pro priori, ubi $n=3$, $\nu=3$ et $\mu=-1$,

$$\int x^{m+2}dx(1-x^3)^{-\frac{1}{3}} = \frac{m}{m+2}\int x^{m-1}dx(1-x^3)^{-\frac{1}{3}}$$

et pro posteriori, ubi $n=3$, $\nu=3$ et $\mu=-2$,

$$\int x^{m+2}dx(1-x^3)^{-\frac{2}{3}} = \frac{m}{m+1}\int x^{m-1}dx(1-x^3)^{-\frac{2}{3}};$$

hinc obtinemus pro forma priori

$$\int \frac{dx}{\sqrt[3]{(1-x^3)}} = A \qquad \int \frac{x\,dx}{\sqrt[3]{(1-x^3)}} = B \qquad \int \frac{xx\,dx}{\sqrt[3]{(1-x^3)}} = C$$

$$\int \frac{x^3dx}{\sqrt[3]{(1-x^3)}} = \frac{1}{3}A \qquad \int \frac{x^4dx}{\sqrt[3]{(1-x^3)}} = \frac{2}{4}B \qquad \int \frac{x^5dx}{\sqrt[3]{(1-x^3)}} = \frac{3}{5}C$$

$$\int \frac{x^6dx}{\sqrt[3]{(1-x^3)}} = \frac{1\cdot 4}{3\cdot 6}A \qquad \int \frac{x^7dx}{\sqrt[3]{(1-x^3)}} = \frac{2\cdot 5}{4\cdot 7}B \qquad \int \frac{x^8dx}{\sqrt[3]{(1-x^3)}} = \frac{3\cdot 6}{5\cdot 8}C$$

$$\int \frac{x^9dx}{\sqrt[3]{(1-x^3)}} = \frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9}A \qquad \int \frac{x^{10}dx}{\sqrt[3]{(1-x^3)}} = \frac{2\cdot 5\cdot 8}{4\cdot 7\cdot 10}B \qquad \int \frac{x^{11}dx}{\sqrt[3]{(1-x^3)}} = \frac{3\cdot 6\cdot 9}{5\cdot 8\cdot 11}C$$

$$\int \frac{x^{12}dx}{\sqrt[3]{(1-x^3)}} = \frac{1\cdot 4\cdot 7\cdot 10}{3\cdot 6\cdot 9\cdot 12}A \qquad \int \frac{x^{13}dx}{\sqrt[3]{(1-x^3)}} = \frac{2\cdot 5\cdot 8\cdot 11}{4\cdot 7\cdot 10\cdot 13}B \qquad \int \frac{x^{14}dx}{\sqrt[3]{(1-x^3)}} = \frac{3\cdot 6\cdot 9\cdot 12}{5\cdot 8\cdot 11\cdot 14}C$$

etc.,

at pro forma posteriori

$$\int\frac{dx}{\sqrt[3]{(1-x^3)^2}}=A' \qquad \int\frac{xdx}{\sqrt[3]{(1-x^3)^2}}=B' \qquad \int\frac{xxdx}{\sqrt[3]{(1-x^3)^2}}=C'$$

$$\int\frac{x^3dx}{\sqrt[3]{(1-x^3)^2}}=\frac{1}{2}A' \qquad \int\frac{x^4dx}{\sqrt[3]{(1-x^3)^2}}=\frac{2}{3}B' \qquad \int\frac{x^5dx}{\sqrt[3]{(1-x^3)^2}}=\frac{3}{4}C'$$

$$\int\frac{x^6dx}{\sqrt[3]{(1-x^3)^2}}=\frac{1\cdot 4}{2\cdot 5}A' \qquad \int\frac{x^7dx}{\sqrt[3]{(1-x^3)^2}}=\frac{2\cdot 5}{3\cdot 6}B' \qquad \int\frac{x^8dx}{\sqrt[3]{(1-x^3)^2}}=\frac{3\cdot 6}{4\cdot 7}C'$$

$$\int\frac{x^9dx}{\sqrt[3]{(1-x^3)^2}}=\frac{1\cdot 4\cdot 7}{2\cdot 5\cdot 8}A' \qquad \int\frac{x^{10}dx}{\sqrt[3]{(1-x^3)^2}}=\frac{2\cdot 5\cdot 8}{3\cdot 6\cdot 9}B' \qquad \int\frac{x^{11}dx}{\sqrt[3]{(1-x^3)^2}}=\frac{3\cdot 6\cdot 9}{4\cdot 7\cdot 10}C'$$

$$\int\frac{x^{12}dx}{\sqrt[3]{(1-x^3)^2}}=\frac{1\cdot 4\cdot 7\cdot 10}{2\cdot 5\cdot 8\cdot 11}A' \qquad \int\frac{x^{13}dx}{\sqrt[3]{(1-x^3)^2}}=\frac{2\cdot 5\cdot 8\cdot 11}{3\cdot 6\cdot 9\cdot 12}B' \qquad \int\frac{x^{14}dx}{\sqrt[3]{(1-x^3)^2}}=\frac{3\cdot 6\cdot 9\cdot 12}{4\cdot 7\cdot 10\cdot 13}C'$$

etc.,

unde concludimus fore generaliter

$$\int\frac{x^{3n}dx}{\sqrt[3]{(1-x^3)}}=\frac{1\cdot 4\cdot 7\cdots(3n-2)}{3\cdot 6\cdot 9\cdots\cdots 3n}A \qquad \int\frac{x^{3n}dx}{\sqrt[3]{(1-x^3)^2}}=\frac{1\cdot 4\cdot 7\cdots(3n-2)}{2\cdot 5\cdot 8\cdots(3n-1)}A'$$

$$\int\frac{x^{3n+1}dx}{\sqrt[3]{(1-x^3)}}=\frac{2\cdot 5\cdot 8\cdots(3n-1)}{4\cdot 7\cdot 10\cdots(3n+1)}B \qquad \int\frac{x^{3n+1}dx}{\sqrt[3]{(1-x^3)^2}}=\frac{2\cdot 5\cdot 8\cdots(3n-1)}{3\cdot 6\cdot 9\cdots\cdots 3n}B'$$

$$\int\frac{x^{3n+2}dx}{\sqrt[3]{(1-x^3)}}=\frac{3\cdot 6\cdot 9\cdots\cdots 3n}{5\cdot 8\cdot 11\cdots(3n+2)}C \qquad \int\frac{x^{3n+2}dx}{\sqrt[3]{(1-x^3)^2}}=\frac{3\cdot 6\cdot 9\cdots\cdots 3n}{4\cdot 7\cdot 10\cdots(3n+1)}C'$$

notandum autem est esse $C=\frac{1}{2}$ et $C'=1$.

COROLLARIUM 1

342. Hae formulae variis modis combinari possunt, ut egregia theoremata inde oriantur; erit scilicet

$$\int\frac{x^{3n}dx}{\sqrt[3]{(1-x^3)}}\cdot\int\frac{x^{3n+2}dx}{\sqrt[3]{(1-x^3)^2}}=\frac{AC'}{3n+1}=\frac{1}{3n+1}\int\frac{dx}{\sqrt[3]{(1-x^3)}},$$

$$\int\frac{x^{3n+1}dx}{\sqrt[3]{(1-x^3)}}\cdot\int\frac{x^{3n}dx}{\sqrt[3]{(1-x^3)^2}}=\frac{A'B}{3n+1}=\frac{1}{3n+1}\int\frac{dx}{\sqrt[3]{(1-x^3)^2}}\cdot\int\frac{xdx}{\sqrt[3]{(1-x^3)}},$$

$$\int\frac{x^{3n+2}dx}{\sqrt[3]{(1-x^3)}}\cdot\int\frac{x^{3n+1}dx}{\sqrt[3]{(1-x^3)^2}}=\frac{2B'C}{3n+2}=\frac{1}{3n+2}\int\frac{xdx}{\sqrt[3]{(1-x^3)^2}}.$$

COROLLARIUM 2

343. Quia nunc ratio exponentium ad ternarium non amplius in computum ingreditur, erit generaliter

$$\int \frac{x^{\lambda-1}dx}{\sqrt[3]{(1-x^3)}} \cdot \int \frac{x^{\lambda+1}dx}{\sqrt[3]{(1-x^3)^2}} = \frac{1}{\lambda} \int \frac{dx}{\sqrt[3]{(1-x^3)}},$$

$$\int \frac{x^{\lambda}dx}{\sqrt[3]{(1-x^3)}} \cdot \int \frac{x^{\lambda-1}dx}{\sqrt[3]{(1-x^3)^2}} = \frac{1}{\lambda} \int \frac{xdx}{\sqrt[3]{(1-x^3)}} \cdot \int \frac{dx}{\sqrt[3]{(1-x^3)^2}},$$

$$\int \frac{x^{\lambda}dx}{\sqrt[3]{(1-x^3)}} \cdot \int \frac{x^{\lambda-1}dx}{\sqrt[3]{(1-x^3)^2}} = \frac{1}{\lambda} \int \frac{xdx}{\sqrt[3]{(1-x^3)^2}},$$

quare ex binis postremis consequimur

$$\int \frac{xdx}{\sqrt[3]{(1-x^3)}} \cdot \int \frac{dx}{\sqrt[3]{(1-x^3)^2}} = \int \frac{xdx}{\sqrt[3]{(1-x^3)^2}}.$$ [1]

COROLLARIUM 3

344. Ponatur $x = z^n$ et $\lambda n = m$ et nostra theoremata sequentes induent ormas

$$\int \frac{z^{m-1}dz}{\sqrt[3]{(1-z^{3n})}} \cdot \int \frac{z^{m+2n-1}dz}{\sqrt[3]{(1-z^{3n})^2}} = \frac{1}{m} \int \frac{z^{n-1}dz}{\sqrt[3]{(1-z^{3n})}},$$

$$\int \frac{z^{m+n-1}dz}{\sqrt[3]{(1-z^{3n})}} \cdot \int \frac{z^{m-1}dz}{\sqrt[3]{(1-z^{3n})^2}} = \frac{n}{m} \int \frac{z^{2n-1}dz}{\sqrt[3]{(1-z^{3n})}} \cdot \int \frac{z^{n-1}dz}{\sqrt[3]{(1-z^{3n})^2}} = \frac{1}{m} \int \frac{z^{2n-1}dz}{\sqrt[3]{(1-z^{3n})^2}}.$$

PROBLEMA 41

345. *Dato integrali* $\int \frac{x^{m-1}dx}{(1-x^n)^{\frac{n-k}{n}}}$ *assignare integrale huius formulae* $\int \frac{x^{m+\lambda n-1}dx}{(1-x^n)^{\frac{n-k}{n}}}$ *posito* $x = 1$.

SOLUTIO

Ut integrale sit finitum, necesse est, ut m et k sint numeri positivi. Cum igitur per reductionem generalem sit

$$\int x^{m+n-1}dx(1-x^n)^{\frac{\mu}{\nu}} = \frac{m\nu}{m\nu + n(\mu+\nu)} \int x^{m-1}dx(1-x^n)^{\frac{\mu}{\nu}},$$

1) Infra (§ 353) demonstrabitur esse $\int \frac{xdx}{\sqrt[3]{(1-x^3)^2}} = \frac{2\pi}{3\sqrt{3}}$ sumto $x = 1$. L. S.

ponatur $\nu = n$ et $\mu = k - n$, ut sit $\mu + \nu = k$; erit

$$\int \frac{x^{m+n-1}dx}{(1-x^n)^{\frac{n-k}{n}}} = \frac{m}{m+k} \int \frac{x^{m-1}dx}{(1-x^n)^{\frac{n-k}{n}}}.$$

Ponatur ergo huius formulae valor, quia datur, $= A$ haecque reductio repetita continuo dabit posito brevitatis gratia P pro $(1-x^n)^{\frac{n-k}{n}}$

$$\int \frac{x^{m-1}dx}{P} = A,$$

$$\int \frac{x^{m+n-1}dx}{P} = \frac{m}{m+k}A,$$

$$\int \frac{x^{m+2n-1}dx}{P} = \frac{m(m+n)}{(m+k)(m+n+k)}A,$$

$$\int \frac{x^{m+3n-1}dx}{P} = \frac{m(m+n)(m+2n)}{(m+k)(m+n+k)(m+2n+k)}A$$

et generaliter

$$\int \frac{x^{m+\alpha n-1}dx}{P} = \frac{m(m+n)(m+2n)\cdots(m+(\alpha-1)n)}{(m+k)(m+n+k)(m+2n+k)\cdots(m+(\alpha-1)n+k)}A.$$

COROLLARIUM 1

346. Si simili modo alia formula sit

$$\int \frac{x^{p-1}dx}{(1-x^n)^{\frac{n-q}{n}}} = B$$

posito $x = 1$, at brevitatis gratia scribatur Q pro $(1-x^n)^{\frac{n-q}{n}}$, habebimus

$$\int \frac{x^{p+\alpha n-1}dx}{Q} = \frac{p(p+n)(p+2n)\cdots(p+(\alpha-1)n)}{(p+q)(p+n+q)(p+2n+q)\cdots(p+(\alpha-1)n+q)}B,$$

quae totidem atque illa continet factores.

COROLLARIUM 2

347. Statuatur nunc $p = m + k$, ut posterior numerator aequalis fiat priori denominatori, et productum harum duarum formularum est

$$\frac{m(m+n)(m+2n)\cdots(m+(\alpha-1)n)}{(m+k+q)(m+n+k+q)(m+2n+k+q)\cdots(m+(\alpha-1)n+k+q)}AB;$$

fiat porro $m+k+q=m+n$ seu $q=n-k$; erit hoc productum $=\frac{m}{m+\alpha n}AB$ ideoque

$$\int\frac{x^{m+\alpha n-1}dx}{(1-x^n)^{\frac{n-k}{n}}}\cdot\int\frac{x^{m+k+\alpha n-1}dx}{(1-x^n)^{\frac{k}{n}}}=\frac{m}{m+\alpha n}\int\frac{x^{m-1}dx}{(1-x^n)^{\frac{n-k}{n}}}\cdot\int\frac{x^{m+k-1}dx}{(1-x^n)^{\frac{k}{n}}},$$

quod est theorema omni attentione dignum, cum hic non amplius opus sit, ut α sit numerus integer.[1])

COROLLARIUM 3

348. Quare loco $m+\alpha n$ scribamus μ; erit

$$\mu\int\frac{x^{\mu-1}dx}{(1-x^n)^{\frac{n-k}{n}}}\int\frac{x^{\mu+k-1}dx}{(1-x^n)^{\frac{k}{n}}}=m\int\frac{x^{m-1}dx}{(1-x^n)^{\frac{n-k}{n}}}\cdot\int\frac{x^{m+k-1}dx}{(1-x^n)^{\frac{k}{n}}}.$$

Hinc si sumamus $m+k=n$ seu $m=n-k$, ob

$$\int\frac{x^{n-1}dx}{(1-x^n)^{\frac{k}{n}}}=\frac{1-(1-x^n)^{\frac{n-k}{n}}}{n-k}=\frac{1}{n-k}$$

posito $x=1$ erit [§ 352]

$$\int\frac{x^{\mu-1}dx}{(1-x^n)^{\frac{n-k}{n}}}\cdot\int\frac{x^{\mu+k-1}dx}{(1-x^n)^{\frac{k}{n}}}=\frac{1}{\mu}\int\frac{x^{n-k-1}dx}{(1-x^n)^{\frac{n-k}{n}}}=\frac{\pi}{\mu n\sin.\frac{k\pi}{n}}.$$

Ac posito $x=z^\nu$, tum vero $\mu\nu=p$, $\nu n=q$ et $k=\lambda n$ habebitur

$$\int\frac{z^{p-1}dz}{(1-z^q)^{1-\lambda}}\cdot\int\frac{z^{p+\lambda q-1}dz}{(1-z^q)^{\lambda}}=\frac{\nu}{p}\int\frac{z^{(1-\lambda)q-1}dz}{(1-z^q)^{1-\lambda}}.{}^{2)}$$

1) Hoc e formula $\Gamma(x+1)=x\Gamma(x)$ (cf. notam p. 209) sponte sequitur. L. S.

2) Editio princeps: $\cdots=\frac{1}{p}\int\frac{z^{(1-\lambda)q-1}dz}{(1-z^q)^{1-\lambda}}$. Correxit L. S.

SCHOLION 1

349. Theoremata particularia, quae hinc consequuntur, ita se habebunt:

$$\text{I.}\quad n=2,\ k=1;\quad \int\frac{x^{\mu-1}dx}{\sqrt{(1-xx)}}\cdot\int\frac{x^{\mu}dx}{\sqrt{(1-xx)}}=\frac{1}{\mu}\int\frac{dx}{\sqrt{(1-xx)}}=\frac{\pi}{2\mu}$$

$$\text{II.}\quad n=3,\ k=1;\quad \int\frac{x^{\mu-1}dx}{\sqrt[3]{(1-x^3)^2}}\cdot\int\frac{x^{\mu}dx}{\sqrt[3]{(1-x^3)}}=\frac{1}{\mu}\int\frac{xdx}{\sqrt[3]{(1-x^3)^2}}=\frac{2\pi}{3\mu\sqrt{3}}$$

$$n=3,\ k=2;\quad \int\frac{x^{\mu-1}dx}{\sqrt[3]{(1-x^3)}}\cdot\int\frac{x^{\mu+1}dx}{\sqrt[3]{(1-x^3)^2}}=\frac{1}{\mu}\int\frac{dx}{\sqrt[3]{(1-x^3)}}=\frac{2\pi}{3\mu\sqrt{3}}$$

$$\text{III.}\quad n=4,\ k=1;\quad \int\frac{x^{\mu-1}dx}{\sqrt[4]{(1-x^4)^3}}\cdot\int\frac{x^{\mu}dx}{\sqrt[4]{(1-x^4)}}=\frac{1}{\mu}\int\frac{xxdx}{\sqrt[4]{(1-x^4)^3}}=\frac{\pi}{2\mu\sqrt{2}}$$

$$n=4,\ k=2;\quad \int\frac{x^{\mu-1}dx}{\sqrt{(1-x^4)}}\cdot\int\frac{x^{\mu+1}dx}{\sqrt{(1-x^4)}}=\frac{1}{\mu}\int\frac{xdx}{\sqrt{(1-x^4)}}=\frac{\pi}{4\mu}$$

$$n=4,\ k=3;\quad \int\frac{x^{\mu-1}dx}{\sqrt[4]{(1-x^4)}}\cdot\int\frac{x^{\mu+2}dx}{\sqrt[4]{(1-x^4)^3}}=\frac{1}{\mu}\int\frac{dx}{\sqrt[4]{(1-x^4)}}=\frac{\pi}{2\mu\sqrt{2}}$$

etc.

Ubi notandum est formulam $\int\frac{x^{n-k-1}dx}{(1-x^n)^{\frac{n-k}{n}}}$ ad rationalitatem reduci posse. Ponatur enim $\frac{x^n}{1-x^n}=z^n$ seu $x^n=\frac{z^n}{1+z^n}$, unde $\frac{dx}{x}=\frac{dz}{z(1+z^n)}$. Quare cum formula nostra sit $=\int\left(\frac{x^n}{1-x^n}\right)^{\frac{n-k}{n}}\cdot\frac{dx}{x}$, evadet ea $=\int\frac{z^{n-k-1}dz}{1+z^n}$, cuius integrale ita determinari debet, ut evanescat posito $x=0$ ideoque $z=0$; tum vero posito $x=1$, hoc est $z=\infty$, dabit valorem, quo hic utimur. Mox [§ 352] autem ostendemus valorem huius integralis $\int\frac{z^{n-k-1}dz}{1+z^n}$ posito $z=\infty$ ideoque et huius $\int\frac{x^{n-k-1}dx}{(1-x^n)^{\frac{n-k}{n}}}$ per angulos exprimi posse, quorum valores hic statim apposui. Deinde etiam notari meretur formulae $\int\frac{x^{m-1}dx}{(1-x^n)^{\frac{n-k}{n}}}$ haec transformatio oriunda posito $1-x^n=z^n$, quae praebet $-\int\frac{z^{k-1}dz}{(1-z^n)^{\frac{n-m}{n}}}$ ita integranda, ut evanescat posito $x=0$ seu $z=1$; tum vero statui debet $x=1$ seu $z=0$. Quod eodem redit, ac si mutato signo haec formula $\int \frac{z^{k-1}dz}{(1-z^n)^{\frac{n-m}{n}}}$ ita integretur, ut evanescat posito $z=0$, tum vero ponatur $z=1$. Cum iam nihil impediat, quominus loco z scribamus x, habebimus hoc insigne theorema

$$\int \frac{x^{m-1}dx}{(1-x^n)^{\frac{n-k}{n}}} = \int \frac{x^{k-1}dx}{(1-x^n)^{\frac{n-m}{n}}},$$

ita ut in huiusmodi formula exponentes m et k inter se commutare liceat, pro casu scilicet $x=1$. Ita pro praecedente formula ad rationalitatem reducibili, ubi $m=n-k$, erit

$$\int \frac{x^{n-k-1}dx}{(1-x^n)^{\frac{n-k}{n}}} = \int \frac{x^{k-1}dx}{(1-x^n)^{\frac{k}{n}}},$$

unde sequitur etiam fore posito $z=\infty$

$$\int \frac{z^{n-k-1}dz}{1+z^n} = \int \frac{z^{k-1}dz}{1+z^n}.$$

SCHOLION 2

350. Hinc etiam formularum magis compositarum integralia pro casu $x=1$ per series concinnas exprimi possunt. Cum enim in reductione superiori posito $m+k=\mu$ seu $k=\mu-m$ sit

$$\int \frac{x^{m+n-1}dx}{(1-x^n)^{\frac{m+n-\mu}{n}}} = \frac{m}{\mu}\int \frac{x^{m-1}dx}{(1-x^n)^{\frac{m+n-\mu}{n}}},$$

si habeatur huiusmodi formula differentialis

$$dy = \frac{x^{m-1}dx}{(1-x^n)^{\frac{m+n-\mu}{n}}}(A + Bx^n + Cx^{2n} + Dx^{3n} + \text{etc.}),$$

quam ita integrari oporteat, ut y evanescat posito $x=0$, ac requiratur valor ipsius y casu $x=1$, erit, si hoc casu fieri ponamus

$$\int \frac{x^{m-1}dx}{(1-x^n)^{\frac{m+n-\mu}{n}}} = O,$$

iste valor

$$= O\left(A + \frac{m}{\mu}B + \frac{m(m+n)}{\mu(\mu+n)}C + \frac{m(m+n)(m+2n)}{\mu(\mu+n)(\mu+2n)}D + \text{etc.}\right).$$

Vicissim ergo proposita hac serie

$$A + \frac{m}{\mu}B + \frac{m(m+n)}{\mu(\mu+n)}C + \frac{m(m+n)(m+2n)}{\mu(\mu+n)(\mu+2n)}D + \text{etc.}$$

eius summa aequabitur huic formulae integrali

$$\frac{1}{O}\int \frac{x^{m-1}dx}{(1-x^n)^{\frac{m+n-\mu}{n}}}(A + Bx^n + Cx^{2n} + Dx^{3n} + \text{etc.}),$$

si post integrationem ponatur $x = 1$. Quodsi ergo eveniat, ut huius seriei $A + Bx^n + Cx^{2n} + \text{etc.}$ summa assignari indeque integratio absolvi queat, obtinebitur summa illius seriei.

PROBLEMA 42

351. *Integralis huius formulae* $\frac{x^{m-1}dx}{1+x^n}$ *ita determinatum, ut posito* $x = 0$ *evanescat, valorem casu* $x = \infty$ *assignare.*

SOLUTIO

Huius formulae integrale iam supra § 77 exhibuimus et quidem ita determinatum, ut posito $x = 0$ evanescat, quod posito brevitatis gratia $\frac{\pi}{n} = \omega$ ita se habet:

$$-\frac{2}{n}\cos. m\omega\, l\sqrt{(1 - 2x\cos.\omega + xx)} + \frac{2}{n}\sin. m\omega \cdot \text{Arc. tang.}\frac{x\sin.\omega}{1 - x\cos.\omega}$$

$$-\frac{2}{n}\cos. 3m\omega\, l\sqrt{(1 - 2x\cos. 3\omega + xx)} + \frac{2}{n}\sin. 3m\omega \cdot \text{Arc. tang.}\frac{x\sin. 3\omega}{1 - x\cos. 3\omega}$$

$$-\frac{2}{n}\cos. 5m\omega\, l\sqrt{(1 - 2x\cos. 5\omega + xx)} + \frac{2}{n}\sin. 5m\omega \cdot \text{Arc. tang.}\frac{x\sin. 5\omega}{1 - x\cos. 5\omega}$$

$$\vdots$$

$$-\frac{2}{n}\cos. \lambda m\omega\, l\sqrt{(1 - 2x\cos. \lambda\omega + xx)} + \frac{2}{n}\sin. \lambda m\omega \cdot \text{Arc. tang.}\frac{x\sin. \lambda\omega}{1 - x\cos. \lambda\omega},$$

ubi λ denotat maximum numerum imparem exponente n minorem, ac si n fuerit ipse numerus impar, insuper accedit pars $\pm\frac{1}{n}l(1+x)$, prout m fuerit vel numerus impar vel par; illo scilicet casu signum $+$, hoc vero signum $-$ valet. Hic igitur quaeritur istius integralis valor, qui prodit posito $x=\infty$. Primo ergo partes logarithmos implicantes expendamus, et quia ob $x=\infty$ est

$$l\sqrt{(1-2x\cos.\lambda\omega+xx)}=l(x-\cos.\lambda\omega)=lx+l\left(1-\frac{\cos.\lambda\omega}{x}\right)=lx$$

ob $\frac{\cos.\lambda\omega}{x}=0$, unde partes logarithmicae praebent

$$-\frac{2lx}{n}(\cos.m\omega+\cos.3m\omega+\cos.5m\omega+\cdots+\cos.\lambda m\omega)\ \left(\pm\frac{lx}{n},\ \text{si } n \text{ impar}\right),$$

ponamus hanc seriem cosinuum

$$\cos.m\omega+\cos.3m\omega+\cos.5m\omega+\cdots+\cos.\lambda m\omega=s$$

eritque per $2\sin.m\omega$ multiplicando

$$\begin{aligned}2s\sin.m\omega=\sin.2m\omega+\sin.4m\omega+\sin.6m\omega+\cdots+\sin.(\lambda+1)m\omega,\\ -\sin.2m\omega-\sin.4m\omega-\sin.6m\omega-\cdots\end{aligned}$$

unde fit

$$s=\frac{\sin.(\lambda+1)m\omega}{2\sin.m\omega}.$$

Quare si n sit numerus par, erit $\lambda=n-1$ sicque partes logarithmicae fiunt

$$-\frac{lx}{n}\cdot\frac{\sin.nm\omega}{\sin.m\omega}=-\frac{lx}{n}\cdot\frac{\sin.m\pi}{\sin.m\omega}$$

ob $n\omega=\pi$. At propter m numerum integrum est $\sin.m\pi=0$, unde hae partes evanescunt. Sin autem sit n numerus impar, est $\lambda=n-2$ et summa partium logarithmicarum fit

$$-\frac{lx}{n}\cdot\frac{\sin.(n-1)m\omega}{\sin.m\omega}\pm\frac{lx}{n};$$

at $\sin.(n-1)m\omega=\sin.(m\pi-m\omega)=\pm\sin.m\omega$, ubi signum superius valet, si m sit numerus impar, contra vero inferius, quod idem de altera ambiguitate est tenendum, ita ut habeamus

$$\mp\frac{lx}{n}\cdot\frac{\sin.m\omega}{\sin.m\omega}\pm\frac{lx}{n}=0.$$

Perpetuo ergo partes logarithmicae se mutuo tollunt; quod etiam inde est perspicuum, quod alioquin integrale foret infinitum, cum tamen manifesto debeat esse finitum.

Relinquuntur ergo soli anguli, quos in unam summam colligamus; consideretur ergo Arc. tang. $\frac{x \sin. \lambda\omega}{1 - x \cos. \lambda\omega}$, qui arcus casu $x = 0$ evanescit, tum vero casu $x = \frac{1}{\cos. \lambda\omega}$ fit quadrans, ulterius ergo aucta x quadrantem superabit, donec facto $x = \infty$ eius tangens fiat $= -\frac{\sin. \lambda\omega}{\cos. \lambda\omega} = -\text{tang.}\, \lambda\omega = \text{tang.}\, (\pi - \lambda\omega)$ ideoque ipse arcus $= \pi - \lambda\omega$, ex quo hi arcus iunctim sumti dabunt

$$\frac{2}{n}\big((\pi-\omega) \sin. m\omega + (\pi-3\omega) \sin. 3m\omega + (\pi-5\omega) \sin. 5m\omega + \cdots + (\pi-\lambda\omega) \sin. \lambda m\omega\big),$$

unde duas series adipiscimur

$$\frac{2\pi}{n}(\sin. m\omega + \sin. 3m\omega + \sin. 5m\omega + \cdots + \sin. \lambda m\omega) = \frac{2\pi}{n} p,$$

$$\frac{-2\omega}{n}(\sin. m\omega + 3 \sin. 3m\omega + 5 \sin. 5m\omega + \cdots + \lambda \sin. \lambda m\omega) = \frac{-2\omega}{n} q,$$

quas seorsim investigemus. Ac pro posteriori quidem cum ante habuissemus

$$\cos. m\omega + \cos. 3m\omega + \cos. 5m\omega + \cdots + \cos. \lambda m\omega = s = \frac{\sin. (\lambda+1) m\omega}{2 \sin. m\omega},$$

si angulum ω ut variabilem spectemus, differentiatio praebet

$$-m d\omega (\sin. m\omega + 3 \sin. 3m\omega + 5 \sin. 5m\omega + \cdots + \lambda \sin. \lambda m\omega)$$
$$= \frac{(\lambda+1) m d\omega \cos. (\lambda+1) m\omega}{2 \sin. m\omega} - \frac{m d\omega \sin. (\lambda+1) m\omega \cos. m\omega}{2 \sin. m\omega^2};$$

ergo

$$-q = \frac{(\lambda+1) \cos. (\lambda+1) m\omega}{2 \sin. m\omega} - \frac{\sin. (\lambda+1) m\omega \cos. m\omega}{2 \sin. m\omega^2}$$

seu

$$-q = \frac{\lambda \cos. (\lambda+1) m\omega}{2 \sin. m\omega} - \frac{\sin. \lambda m\omega}{2 \sin. m\omega^2}.$$

Pro altera serie

$$p = \sin. m\omega + \sin. 3m\omega + \sin. 5m\omega + \cdots + \sin. \lambda m\omega$$

multiplicemus utrinque per $2 \sin. m\omega$ fietque

$$2p\sin.m\omega = 1 - \cos.2m\omega - \cos.4m\omega - \cos.6m\omega - \cdots - \cos.(\lambda+1)m\omega$$
$$+ \cos.2m\omega + \cos.4m\omega + \cos.6m\omega + \cdots$$

sicque erit

$$p = \frac{1-\cos.(\lambda+1)m\omega}{2\sin.m\omega}.$$

Quodsi iam fuerit n numerus par, erit $\lambda = n-1$ indeque

$$\cos.(\lambda+1)m\omega = \cos.nm\omega = \cos.m\pi \quad \text{et} \quad \sin.(\lambda+1)m\omega = \sin.m\pi = 0,$$

ergo

$$p = \frac{1-\cos.m\pi}{2\sin.m\omega} \quad \text{et} \quad -q = \frac{n\cos.m\pi}{2\sin.m\omega}$$

hincque omnes arcus iunctim sumti

$$\frac{2\pi}{n}\cdot\frac{1-\cos.m\pi}{2\sin.m\omega} + \frac{2\omega}{n}\cdot\frac{n\cos.m\pi}{2\sin.m\omega} = \frac{\pi}{n\sin.m\omega}$$

ob $n\omega = \pi$.

Sit nunc n numerus impar; erit $\lambda = n-2$ indeque

$$\cos.(\lambda+1)m\omega = \cos.(m\pi - m\omega) \quad \text{et} \quad \sin.(\lambda+1)m\omega = \sin.(m\pi - m\omega)$$

seu

$$\cos.(\lambda+1)m\omega = \cos.m\pi\cos.m\omega \quad \text{et} \quad \sin.(\lambda+1)m\omega = -\cos.m\pi\sin.m\omega,$$

ergo

$$p = \frac{1-\cos.m\pi\cos.m\omega}{2\sin.m\omega} \quad \text{et} \quad -q = \frac{(n-1)\cos.m\pi\cos.m\omega}{2\sin.m\omega} + \frac{\cos.m\pi\cos.m\omega}{2\sin.m\omega},$$

unde summa omnium angulorum

$$\frac{\pi(1-\cos.m\pi\cos.m\omega)}{n\sin.m\omega} + \frac{\omega(n-1)\cos.m\pi\cos.m\omega}{n\sin.m\omega} + \frac{\omega\cos.m\pi\cos.m\omega}{n\sin.m\omega},$$

quae ob $n\omega = \pi$ reducitur ad $\frac{\pi}{n\sin.m\omega}$.

Sive ergo exponens n sit par sive impar, posito $x = \infty$ habemus

$$\int\frac{x^{m-1}dx}{1+x^n} = \frac{\pi}{n\sin.m\omega} = \frac{\pi}{n\sin.\frac{m\pi}{n}}.$$

COROLLARIUM 1

352. Hinc ergo erit formula supra memorata (§ 349)

$$\int\frac{z^{n-k-1}dz}{1+z^n}=\int\frac{z^{k-1}dz}{1+z^n}=\frac{\pi}{n\sin.\frac{(n-k)\pi}{n}}=\frac{\pi}{n\sin.\frac{k\pi}{n}}$$

posito $z=\infty$. Unde sequitur fore etiam formulam, cui hanc aequari ostendimus,

$$\int\frac{x^{n-k-1}dx}{(1-x^n)^{\frac{n-k}{n}}}=\int\frac{x^{k-1}dx}{(1-x^n)^{\frac{k}{n}}}=\frac{\pi}{n\sin.\frac{k\pi}{n}}$$

posito $x=1$.

COROLLARIUM 2

353. Percurramus casus simpliciores pro utroque formularum genere posito $z=\infty$ et $x=1$:

$$\int\frac{dz}{1+zz}=\int\frac{dx}{\sqrt{(1-xx)}}=\frac{\pi}{2\sin.\frac{1}{2}\pi}=\frac{\pi}{2},$$

$$\int\frac{dz}{1+z^3}=\int\frac{zdz}{1+z^3}=\int\frac{dx}{\sqrt[3]{(1-x^3)}}=\int\frac{xdx}{\sqrt[3]{(1-x^3)^2}}=\frac{\pi}{3\sin.\frac{1}{3}\pi}=\frac{2\pi}{3\sqrt{3}},$$

$$\int\frac{dz}{1+z^4}=\int\frac{zzdz}{1+z^4}=\int\frac{dx}{\sqrt[4]{(1-x^4)}}=\int\frac{xxdx}{\sqrt[4]{(1-x^4)^3}}=\frac{\pi}{4\sin.\frac{1}{4}\pi}=\frac{\pi}{2\sqrt{2}},$$

$$\int\frac{dz}{1+z^6}=\int\frac{z^4dz}{1+z^6}=\int\frac{dx}{\sqrt[6]{(1-x^6)}}=\int\frac{x^4dx}{\sqrt[6]{(1-x^6)^5}}=\frac{\pi}{6\sin.\frac{1}{6}\pi}=\frac{\pi}{3}.$$

COROLLARIUM 3

354. Cum sit

$$\frac{1}{(1-x^n)^{\frac{k}{n}}}=1+\frac{k}{n}x^n+\frac{k(k+n)}{n\cdot 2n}x^{2n}+\frac{k(k+n)(k+2n)}{n\cdot 2n\cdot 3n}x^{3n}+\text{etc.},$$

erit per $x^{k-1}dx$ multiplicando, tum integrando ac $x=1$ ponendo

$$\frac{\pi}{n\sin.\frac{k\pi}{n}}=\frac{1}{k}+\frac{k}{n(k+n)}+\frac{k(k+n)}{n\cdot 2n(k+2n)}+\frac{k(k+n)(k+2n)}{n\cdot 2n\cdot 3n(k+3n)}+\text{etc.}$$

et loco k scribendo $n-k$ erit quoque

$$\frac{\pi}{n\sin.\frac{k\pi}{n}}=\frac{1}{n-k}+\frac{n-k}{n(2n-k)}+\frac{(n-k)(2n-k)}{n\cdot 2n(3n-k)}+\frac{(n-k)(2n-k)(3n-k)}{n\cdot 2n\cdot 3n(4n-k)}+\text{etc.}$$

SCHOLION

355. Pro formulis quantitates transcendentes continentibus supra iam praecipuos valores, quos integralia, dum variabili certus quidam valor tribuitur, recipiunt, evolvimus, ita ut non opus sit huiusmodi formulas hic denuo examinare. Hinc autem intelligitur eos valores integralis $\int X dx$ prae reliquis esse notatu dignos ac plerumque multo succinctius exprimi posse, qui eiusmodi valoribus variabilis x respondent, quibus functio X vel fit infinita vel in nihilum abit. Ita integralia formularum $\int \frac{x^{m-1}dx}{(1-x^n)^{\frac{\mu}{\nu}}}$ et $\int \frac{z^{m-1}dz}{1+z^n}$ valores prae reliquis memorabiles recipiunt, si fiat $x = 1$ et $z = \infty$, ubi illius denominator evanescit, huius vero fit infinitus. Caeterum omni attentione dignum est, quod hic ostendimus, formulae integralis $\int \frac{z^{m-1}dz}{1+z^n}$ valorem casu $z = \infty$ tam concinne exprimi, ut sit $\frac{\pi}{n \sin. \frac{m\pi}{n}}$, cuius demonstratio cum per tot ambages sit adstructa, merito suspicionem excitat eam via multo faciliori confici posse, etiamsi modus nondum perspiciatur. Id quidem manifestum est hanc demonstrationem ex ratione sinuum angulorum multiplorum peti oportere; et quoniam in *Introductione*[1]) $\sin. \frac{m\pi}{n}$ per productum infinitorum factorum expressi, mox videbimus inde eandem veritatem multo facilius deduci posse, etiamsi ne hanc quidem viam pro maxime naturali haberi velim.

Sequens autem caput huiusmodi investigationi destinavi, quo valores integralium, quos uti in hoc capite certo quodam casu recipiunt, per producta infinita seu ex innumeris factoribus constantia exprimere docebo; quandoquidem hinc insignia subsidia in Analysin redundant pluraque alia incrementa inde expectari possunt.

1) *Introductio,* t. I cap. XI, § 184; vide etiam notam p. 76. L. S.

CAPUT IX

DE EVOLUTIONE INTEGRALIUM PER PRODUCTA INFINITA

PROBLEMA 43

356. *Valorem huius integralis* $\int \frac{dx}{\sqrt{(1-xx)}}$, *quem casu* $x=1$ *recipit, in productum infinitum evolvere.*

SOLUTIO

Quemadmodum supra formulas altiores ad simplicem reduximus, ita hic formulam $\int \frac{dx}{\sqrt{(1-xx)}}$ continuo ad altiores perducamus. Ita, cum posito $x=1$ sit

$$\int \frac{x^{m-1}dx}{\sqrt{(1-xx)}} = \frac{m+1}{m} \int \frac{x^{m+1}dx}{\sqrt{(1-xx)}},$$

erit

$$\int \frac{dx}{\sqrt{(1-xx)}} = \frac{2}{1} \int \frac{xxdx}{\sqrt{(1-xx)}} = \frac{2 \cdot 4}{1 \cdot 3} \int \frac{x^4dx}{\sqrt{(1-xx)}} = \frac{2 \cdot 4 \cdot 6}{1 \cdot 3 \cdot 5} \int \frac{x^6dx}{\sqrt{(1-xx)}} \quad \text{etc.},$$

unde concludimus fore indefinite

$$\int \frac{dx}{\sqrt{(1-xx)}} = \frac{2 \cdot 4 \cdot 6 \cdot 8 \cdots\cdots 2i}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2i-1)} \int \frac{x^{2i}dx}{\sqrt{(1-xx)}}$$

atque adeo etiam, si pro i sumatur numerus infinitus. Nunc simili modo a formula $\int \frac{xdx}{\sqrt{(1-xx)}}$ ascendamus reperiemusque

$$\int \frac{xdx}{\sqrt{(1-xx)}} = \frac{3 \cdot 5 \cdot 7 \cdot 9 \cdots (2i+1)}{2 \cdot 4 \cdot 6 \cdot 8 \cdots\cdots 2i} \int \frac{x^{2i+1}dx}{\sqrt{(1-xx)}}$$

atque observo, si i sit numerus infinitus, formulas istas

$$\int \frac{x^{2i}\,dx}{\sqrt{(1-xx)}} \quad \text{et} \quad \int \frac{x^{2i+1}\,dx}{\sqrt{(1-xx)}}$$

rationem aequalitatis esse habituras. Ex reductione enim principali perspicuum est, si m sit numerus infinitus, fore

$$\int \frac{x^{m-1}\,dx}{\sqrt{(1-xx)}} = \int \frac{x^{m+1}\,dx}{\sqrt{(1-xx)}} = \int \frac{x^{m+3}\,dx}{\sqrt{(1-xx)}}$$

atque adeo in genere[1])

$$\int \frac{x^{m+\mu}\,dx}{\sqrt{(1-xx)}} = \int \frac{x^{m+\nu}\,dx}{\sqrt{(1-xx)}},$$

quantumvis magna fuerit differentia inter μ et ν, modo finita. Cum igitur sit

$$\int \frac{x^{2i}\,dx}{\sqrt{(1-xx)}} = \int \frac{x^{2i+1}\,dx}{\sqrt{(1-xx)}},$$

si ponamus

$$\frac{2\cdot 4\cdot 6\cdots\cdots 2i}{1\cdot 3\cdot 5\cdots(2i-1)} = M \quad \text{et} \quad \frac{3\cdot 5\cdot 7\cdot 9\cdots(2i+1)}{2\cdot 4\cdot 6\cdot 8\cdots\cdots 2i} = N,$$

erit

$$\int \frac{dx}{\sqrt{(1-xx)}} : \int \frac{x\,dx}{\sqrt{(1-xx)}} = M : N = \frac{M}{N} : 1$$

posito $x = 1$. At est

$$\int \frac{x\,dx}{\sqrt{(1-xx)}} = 1 \quad \text{et} \quad \int \frac{dx}{\sqrt{(1-xx)}} = \frac{\pi}{2},$$

unde colligitur

$$\int \frac{dx}{\sqrt{(1-xx)}} = \frac{M}{N}.$$

Quia producta M et N ex aequali factorum numero constant, si primum factorem $\frac{2}{1}$ producti M per primum factorem $\frac{3}{2}$ producti N, secundum $\frac{4}{3}$ illius per secundum $\frac{5}{4}$ huius et ita porro dividamus, fiet

$$\frac{M}{N} = \frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdot\frac{6\cdot 6}{5\cdot 7}\cdot\frac{8\cdot 8}{7\cdot 9}\cdot\text{etc.},$$

1) Pro casu, quo exponentes ipsius x non binario differunt, vide § 359. L. S.

unde obtinemus pro casu $x = 1$ per productum infinitum

$$\int \frac{dx}{\sqrt{(1-xx)}} = \frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdot\frac{6\cdot 6}{5\cdot 7}\cdot\frac{8\cdot 8}{7\cdot 9}\cdot \text{etc.} = \frac{\pi}{2}\cdot$$

COROLLARIUM 1

357. Pro valore ergo ipsius π idem productum infinitum elicuimus, quod olim iam WALLISIUS invenerat et cuius veritatem in *Introductione*[1]) confirmavimus diversissimis viis incedentes; erit itaque

$$\pi = 2\cdot\frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdot\frac{6\cdot 6}{5\cdot 7}\cdot\frac{8\cdot 8}{7\cdot 9}\cdot \text{etc.}$$

COROLLARIUM 2

358. Nihil interest, quonam ordine singuli factores in hoc producto disponantur, dummodo nulli relinquantur. Ita aliquot ab initio seorsim sumendo reliqui ordine debito disponi possunt, veluti

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2\cdot 4}{3\cdot 3}\cdot\frac{4\cdot 6}{5\cdot 5}\cdot\frac{6\cdot 8}{7\cdot 7}\cdot\frac{8\cdot 10}{9\cdot 9}\cdot \text{etc.}$$

vel

$$\frac{\pi}{2} = \frac{2\cdot 4}{1\cdot 3}\cdot\frac{2\cdot 6}{3\cdot 5}\cdot\frac{4\cdot 8}{5\cdot 7}\cdot\frac{6\cdot 10}{7\cdot 9}\cdot\frac{8\cdot 12}{9\cdot 11}\cdot \text{etc.}$$

vel

$$\frac{\pi}{2} = \frac{2}{3}\cdot\frac{2\cdot 4}{1\cdot 5}\cdot\frac{4\cdot 6}{3\cdot 7}\cdot\frac{6\cdot 8}{5\cdot 9}\cdot\frac{8\cdot 10}{7\cdot 11}\cdot \text{etc.}$$

vel

$$\frac{\pi}{2} = \frac{2\cdot 4}{3\cdot 5}\cdot\frac{2\cdot 6}{1\cdot 7}\cdot\frac{4\cdot 8}{3\cdot 9}\cdot\frac{6\cdot 10}{5\cdot 11}\cdot\frac{8\cdot 12}{7\cdot 13}\cdot \text{etc.}$$

SCHOLION

359. Fundamentum ergo huius evolutionis in hoc consistit, quod valor integralis $\int \frac{x^{i+\alpha}dx}{\sqrt{(1-xx)}}$ denotante i numerum infinitum idem sit, utcunque numerus finitus α varietur. Atque hoc quidem ex reductione

$$\int \frac{x^{i-1}dx}{\sqrt{(1-xx)}} = \frac{i+1}{i}\int \frac{x^{i+1}dx}{\sqrt{(1-xx)}}$$

1) *Introductio*, t. I cap. XI, § 185. L. S.

manifestum est, si pro α valores binario differentes assumantur. Deinde autem nullum est dubium, quin hoc integrale $\int\frac{x^{i+1}dx}{\sqrt{(1-xx)}}$ inter haec $\int\frac{x^{i}dx}{\sqrt{(1-xx)}}$ et $\int\frac{x^{i+2}dx}{\sqrt{(1-xx)}}$ quasi limites contineatur, qui cum sint inter se aequales, necesse est omnes formulas intermedias iisdem quoque esse aequales. Atque hoc latius patet ad formulas magis complicatas, ita ut denotante i numerum infinitum sit

$$\int\frac{x^{i+\alpha}dx}{(1-x^n)^k}=\int\frac{x^{i}dx}{(1-x^n)^k}.$$

Cum enim sit

$$\int\frac{x^{m+n-1}dx}{(1-x^n)^{\frac{n-k}{n}}}=\frac{m}{m+k}\int\frac{x^{m-1}dx}{(1-x^n)^{\frac{n-k}{n}}},$$

hae formulae posito $m=\infty$ sunt aequales; unde illarum quoque aequalitas casibus, quibus $\alpha=n$ vel $\alpha=2n$ vel $\alpha=3n$ etc., perspicitur; sin autem α medium quempiam valorem teneat, formulae ipsius quoque valor medium quoddam tenere debet inter valores aequales ideoque ipsis erit aequalis. Hoc igitur principio stabilito sequens problema resolvere poterimus.

PROBLEMA 44

360. *Rationem horum duorum integralium*

$$\int x^{m-1}dx(1-x^n)^{\frac{k-n}{n}} \quad et \quad \int x^{\mu-1}dx(1-x^n)^{\frac{k-n}{n}}$$

casu $x=1$ *per productum infinitorum factorum exprimere.*

SOLUTIO

Cum sit

$$\int x^{m-1}dx(1-x^n)^{\frac{k-n}{n}}=\frac{m+k}{n}\int x^{m+n-1}dx(1-x^n)^{\frac{k-n}{n}}$$

casu $x=1$, valor istius integralis ad integrale infinite remotum reducetur hoc modo

$$\int x^{m-1}dx(1-x^n)^{\frac{k-n}{n}}$$
$$=\frac{(m+k)(m+k+n)(m+k+2n)\cdots(m+k+in)}{m\quad(m+n)\quad(m+2n)\quad\cdots\quad(m+in)}\int x^{m+in+n-1}dx(1-x^n)^{\frac{k-n}{n}},$$

ubi i numerum infinitum denotare assumimus. Simili autem modo pro altera formula proposita erit

$$\int x^{\mu-1}dx(1-x^n)^{\frac{k-n}{n}}$$

$$=\frac{(\mu+k)(\mu+k+n)(\mu+k+2n)\cdots(\mu+k+in)}{\mu\quad(\mu+n)\quad(\mu+2n)\quad\cdots\quad(\mu+in)}\int x^{\mu+in+n-1}dx(1-x^n)^{\frac{k-n}{n}}$$

atque hae postremae formulae integrales ob exponentes infinitos aequales erunt non obstante inaequalitate numerorum m et μ; tum vero bina haec producta infinita pari factorum numero constant. Quare si singuli per singulos, hoc est primus per primum, secundus per secundum [et ita porro] dividantur, ratio binorum integralium propositorum ita exprimetur

$$\frac{\int x^{m-1}dx(1-x^n)^{\frac{k-n}{n}}}{\int x^{\mu-1}dx(1-x^n)^{\frac{k-n}{n}}}=\frac{\mu(m+k)}{m(\mu+k)}\cdot\frac{(\mu+n)(m+k+n)}{(m+n)(\mu+k+n)}\cdot\frac{(\mu+2n)(m+k+2n)}{(m+2n)(\mu+k+2n)}\cdot\text{etc.},$$

si quidem ambo integralia ita determinentur, ut posito $x=0$ evanescant, tum vero statuatur $x=1$; litteris autem m, μ, n, k numeros positivos denotari necesse est.

COROLLARIUM 1

361. Si differentia numerorum m et μ aequetur multiplo ipsius n, in producto invento infiniti factores se destruunt relinqueturque factorum numerus finitus, uti, si $\mu=m+n$, habebitur

$$\frac{(m+n)(m+k)}{m(m+k+n)}\cdot\frac{(m+2n)(m+k+n)}{(m+n)(m+k+2n)}\cdot\frac{(m+3n)(m+k+2n)}{(m+2n)(m+k+3n)}\cdot\text{etc.},$$

quod reducitur ad $\frac{m+k}{m}$.

COROLLARIUM 2

362. Valor autem illius producti necessario est finitus, id quod tam ex formulis integralibus, quarum rationem exprimit, patet quam inde, quod in singulis factoribus numeratores et denominatores sunt alternatim maiores et minores.

COROLLARIUM 3

363. Si ponamus $m = 1$, $\mu = 3$, $n = 4$ et $k = 2$, erit

$$\frac{\int \frac{dx}{\sqrt{(1-x^4)}}}{\int \frac{xxdx}{\sqrt{(1-x^4)}}} = \frac{3 \cdot 3}{1 \cdot 5} \cdot \frac{7 \cdot 7}{5 \cdot 9} \cdot \frac{11 \cdot 11}{9 \cdot 13} \cdot \frac{15 \cdot 15}{13 \cdot 17} \cdot \text{etc.};$$

supra autem invenimus productum harum binarum formularum esse $= \frac{\pi}{4}$.

PROBLEMA 45

364. *Valorem huius integralis* $\int x^{m-1}dx(1-x^n)^{\frac{k-n}{n}}$, *quem posito* $x = 1$ *recipit, per productum infinitum exprimere.*

SOLUTIO

Cum in problemate praecedente ratio huius integralis ad hoc alterum

$$\int x^{\mu-1}dx(1-x^n)^{\frac{k-n}{n}}$$

per productum infinitum sit assignata, in hoc exponens μ ita accipiatur, ut integrale exhiberi possit. Capiatur ergo $\mu = n$ et integrale fit

$$= C - \frac{1}{k}(1-x^n)^{\frac{k}{n}} = \frac{1-(1-x^n)^{\frac{k}{n}}}{k}$$

ita determinatum, ut posito $x = 0$ evanescat; ponatur nunc, ut conditio postulat, $x = 1$, et quia hoc integrale erit $= \frac{1}{k}$, habebimus formulae propositae integrale casu $x = 1$ ita expressum

$$\int x^{m-1}dx(1-x^n)^{\frac{k-n}{n}} = \frac{1}{k} \cdot \frac{n(m+k)}{m(k+n)} \cdot \frac{2n(m+k+n)}{(m+n)(k+2n)} \cdot \frac{3n(m+k+2n)}{(m+2n)(k+3n)} \cdot \text{etc.},$$

quod singulos factores partiendo ita repraesentari potest

$$\int x^{m-1}dx(1-x^n)^{\frac{k-n}{n}} = \frac{n}{mk} \cdot \frac{2n(m+k)}{(m+n)(k+n)} \cdot \frac{3n(m+k+n)}{(m+2n)(k+2n)} \cdot \frac{4n(m+k+2n)}{(m+3n)(k+3n)} \cdot \text{etc.}$$

COROLLARIUM 1

365. Cum in hac expressione litterae m et k sint permutabiles, sequitur etiam haec integralia posito $x=1$ inter se esse aequalia

$$\int x^{m-1}dx(1-x^n)^{\frac{k-n}{n}} = \int x^{k-1}dx(1-x^n)^{\frac{m-n}{n}},$$

quam aequalitatem iam supra § 349 elicuimus.

COROLLARIUM 2

366. Cum formulae nostrae valor, si $m=n-k$, aequalis sit valori huius $\int\frac{z^{k-1}dz}{1+z^n}$ posito $z=\infty$, si ob $m+k=n$ statuamus $m=\frac{n+\alpha}{2}$ et $k=\frac{n-\alpha}{2}$, habebimus

$$\int\frac{x^{m-1}dx}{(1-x^n)^{\frac{n+\alpha}{2n}}} = \int\frac{x^{k-1}dx}{(1-x^n)^{\frac{n-\alpha}{2n}}} = \int\frac{z^{k-1}dz}{1+z^n} = \int\frac{z^{m-1}dz}{1+z^n}$$

$$= \frac{4n}{nn-\alpha\alpha}\cdot\frac{2\cdot 4nn}{9nn-\alpha\alpha}\cdot\frac{4\cdot 6nn}{25nn-\alpha\alpha}\cdot\frac{6\cdot 8nn}{49nn-\alpha\alpha}\cdot\text{etc.}$$

Quod productum etiam hoc modo exponi potest

$$\frac{2}{n-\alpha}\cdot\frac{2n\cdot 2n}{(n+\alpha)(3n-\alpha)}\cdot\frac{4n\cdot 4n}{(3n+\alpha)(5n-\alpha)}\cdot\frac{6n\cdot 6n}{(5n+\alpha)(7n-\alpha)}\cdot\text{etc.},$$

quod ergo etiam exprimit valorem ipsius $\frac{\pi}{n\sin.\frac{m\pi}{n}} = \frac{\pi}{n\cos.\frac{\alpha\pi}{2n}}$ per § 351.

COROLLARIUM 3

367. Vel si simpliciter ponamus $k=n-m$, fiet

$$\int\frac{x^{m-1}dx}{(1-x^n)^{\frac{m}{n}}} = \int\frac{x^{n-m-1}dx}{(1-x^n)^{\frac{n-m}{n}}} = \int\frac{z^{m-1}dz}{1+z^n} = \int\frac{z^{n-m-1}dz}{1+z^n}$$

$$= \frac{1}{n-m}\cdot\frac{nn}{m(2n-m)}\cdot\frac{4nn}{(n+m)(3n-m)}\cdot\frac{9nn}{(2n+m)(4n-m)}\cdot\text{etc.},$$

quae ex forma primum inventa oritur. Haec ergo aequalitas subsistit, si ponatur $x=1$ et $z=\infty$.

SCHOLION 1

368. In *Introductione*[1]) autem pro multiplicatione angulorum inveneram

$$\sin.\frac{m\pi}{n}=\frac{m\pi}{n}\left(1-\frac{mm}{nn}\right)\left(1-\frac{mm}{4nn}\right)\left(1-\frac{mm}{9nn}\right)\left(1-\frac{mm}{16nn}\right)\cdot\text{etc.},$$

et cum $\sin.\frac{(n-m)\pi}{n}=\sin.\frac{m\pi}{n}$, ob $n-m=k$ erit etiam

$$\sin.\frac{m\pi}{n}=\frac{k\pi}{n}\left(1-\frac{kk}{nn}\right)\left(1-\frac{kk}{4nn}\right)\left(1-\frac{kk}{9nn}\right)\left(1-\frac{kk}{16nn}\right)\cdot\text{etc.},$$

quae reducitur ad hanc formam

$$\sin.\frac{m\pi}{n}=\frac{k\pi}{n}\cdot\frac{(n-k)(n+k)}{nn}\cdot\frac{(2n-k)(2n+k)}{4nn}\cdot\frac{(3n-k)(3n+k)}{9nn}\cdot\text{etc.}$$

et pro k suo valore restituto

$$\sin.\frac{m\pi}{n}=\frac{\pi}{n}(n-m)\cdot\frac{m(2n-m)}{nn}\cdot\frac{(n+m)(3n-m)}{4nn}\cdot\frac{(2n+m)(4n-m)}{9nn}\cdot\text{etc.},$$

unde manifesto pro $\frac{\pi}{n\sin.\frac{m\pi}{n}}$ idem reperitur productum, quod valorem nostrorum integralium exprimit, sicque novam habemus demonstrationem pro theoremate illo eximio supra [§ 351] per multas ambages evicto esse

$$\int\frac{x^{m-1}dx}{(1-x^n)^{\frac{m}{n}}}=\int\frac{x^{n-m-1}dx}{(1-x^n)^{\frac{n-m}{n}}}=\int\frac{z^{m-1}dz}{1+z^n}=\int\frac{z^{n-m-1}dz}{1+z^n}=\frac{\pi}{n\sin.\frac{m\pi}{n}}.$$

SCHOLION 2

369. Quo nostra formula latius pateat, ponamus $\frac{k}{n}=\frac{\mu}{\nu}$ seu $k=\frac{\mu n}{\nu}$ et nanciscemur

$$\int x^{m-1}dx(1-x^n)^{\frac{\mu}{\nu}-1}=\frac{\nu}{m\mu}\cdot\frac{2(m\nu+n\mu)}{(m+n)(\mu+\nu)}\cdot\frac{3(m\nu+n(\mu+\nu))}{(m+2n)(\mu+2\nu)}\cdot\frac{4(m\nu+n(\mu+2\nu))}{(m+3n)(\mu+3\nu)}\cdot\text{etc.}$$

$$=\frac{\nu}{m\mu}\cdot\frac{2(m\nu+n\mu)}{(m+n)(\mu+\nu)}\cdot\frac{3(m\nu+n\mu+n\nu)}{(m+2n)(\mu+2\nu)}\cdot\frac{4(m\nu+n\mu+2n\nu)}{(m+3n)(\mu+3\nu)}\cdot\frac{5(m\nu+n\mu+3n\nu)}{(m+4n)(\mu+4\nu)}\cdot\text{etc.},$$

in qua expressione litterae m, n et μ, ν sunt permutabiles praeterquam in primo factore, qui cum reliquis lege continuitatis non connectitur; ac si per n multiplicemus, permutabilitas erit perfecta, unde concludimus fore

1) *Introductio*, t. I cap. XI, § 184. L. S.

$$n\int x^{m-1}dx(1-x^n)^{\frac{\mu}{\nu}-1} = \nu\int x^{\mu-1}dx(1-x^\nu)^{\frac{m}{n}-1},$$

quae aequalitas casu $\nu = n$ ad supra observatam reducitur. Caeterum iuvabit casus praecipuos perpendisse, quos ex valoribus μ et ν desumamus.

EXEMPLUM 1

370. Sit $\mu = 1$ et $\nu = 2$ fietque

$$\int\frac{x^{m-1}dx}{\sqrt{(1-x^n)}} = \frac{2}{m}\cdot\frac{2(2m+n)}{3(m+n)}\cdot\frac{3(2m+3n)}{5(m+2n)}\cdot\frac{4(2m+5n)}{7(m+3n)}\cdot\text{etc.} = \frac{2}{n}\int\frac{dx}{\sqrt[n]{(1-x^2)^{n-m}}},$$

quae expressio ita commodius repraesentatur

$$\int\frac{x^{m-1}dx}{\sqrt{(1-x^n)}} = \frac{2}{m}\cdot\frac{4(2m+n)}{3(2m+2n)}\cdot\frac{6(2m+3n)}{5(2m+4n)}\cdot\frac{8(2m+5n)}{7(2m+6n)}\cdot\text{etc.},$$

unde sequentes casus specialissimi deducuntur

$$\int\frac{dx}{\sqrt{(1-xx)}} = 2\cdot\frac{2\cdot 4}{3\cdot 3}\cdot\frac{4\cdot 6}{5\cdot 5}\cdot\frac{6\cdot 8}{7\cdot 7}\cdot\text{etc.} = \int\frac{dx}{\sqrt{(1-xx)}},$$

$$\int\frac{dx}{\sqrt{(1-x^3)}} = 2\cdot\frac{4\cdot 5}{3\cdot 8}\cdot\frac{6\cdot 11}{5\cdot 14}\cdot\frac{8\cdot 17}{7\cdot 20}\cdot\frac{10\cdot 23}{9\cdot 26}\cdot\text{etc.} = \frac{2}{3}\int\frac{dx}{\sqrt[3]{(1-x^2)^2}},$$

$$\int\frac{xdx}{\sqrt{(1-x^3)}} = 1\cdot\frac{4\cdot 7}{3\cdot 10}\cdot\frac{6\cdot 13}{5\cdot 16}\cdot\frac{8\cdot 19}{7\cdot 22}\cdot\frac{10\cdot 25}{9\cdot 28}\cdot\text{etc.} = \frac{2}{3}\int\frac{dx}{\sqrt[3]{(1-x^2)}},$$

$$\int\frac{dx}{\sqrt{(1-x^4)}} = 2\cdot\frac{4\cdot 3}{3\cdot 5}\cdot\frac{6\cdot 7}{5\cdot 9}\cdot\frac{8\cdot 11}{7\cdot 13}\cdot\frac{10\cdot 15}{9\cdot 17}\cdot\text{etc.} = \frac{1}{2}\int\frac{dx}{\sqrt[4]{(1-x^2)^3}},$$

$$\int\frac{xdx}{\sqrt{(1-x^4)}} = 1\cdot\frac{4\cdot 4}{3\cdot 6}\cdot\frac{6\cdot 8}{5\cdot 10}\cdot\frac{8\cdot 12}{7\cdot 14}\cdot\frac{10\cdot 16}{9\cdot 18}\cdot\text{etc.} = \frac{1}{2}\int\frac{dx}{\sqrt{(1-x^2)}}$$

sive

$$= 1\cdot\frac{2\cdot 4}{3\cdot 3}\cdot\frac{4\cdot 6}{5\cdot 5}\cdot\frac{6\cdot 8}{7\cdot 7}\cdot\frac{8\cdot 10}{9\cdot 9}\cdot\text{etc.},$$

$$\int\frac{xxdx}{\sqrt{(1-x^4)}} = \frac{2}{3}\cdot\frac{4\cdot 5}{3\cdot 7}\cdot\frac{6\cdot 9}{5\cdot 11}\cdot\frac{8\cdot 13}{7\cdot 15}\cdot\frac{10\cdot 17}{9\cdot 19}\cdot\text{etc.} = \frac{1}{2}\int\frac{dx}{\sqrt[4]{(1-xx)}},$$

$$\int\frac{x^3dx}{\sqrt{(1-x^4)}} = \frac{2}{4}\cdot\frac{4\cdot 6}{3\cdot 8}\cdot\frac{6\cdot 10}{5\cdot 12}\cdot\frac{8\cdot 14}{7\cdot 16}\cdot\frac{10\cdot 18}{9\cdot 20}\cdot\text{etc.} = \frac{1}{2}.$$

EXEMPLUM 2

371. Sit $\mu = 1$ et $\nu = 3$ fietque

$$\int\frac{x^{m-1}dx}{\sqrt[3]{(1-x^n)^2}} = \frac{3}{m}\cdot\frac{2(3m+n)}{4(m+n)}\cdot\frac{3(3m+4n)}{7(m+2n)}\cdot\frac{4(3m+7n)}{10(m+3n)}\cdot\text{etc.} = \frac{3}{n}\int\frac{dx}{\sqrt[n]{(1-x^3)^{n-m}}},$$

unde sequentes casus specialissimi deducuntur

$$\int\frac{dx}{\sqrt[3]{(1-x^2)^2}}=\frac{3}{1}\cdot\frac{2\cdot5}{4\cdot3}\cdot\frac{3\cdot11}{7\cdot5}\cdot\frac{4\cdot17}{10\cdot7}\cdot\frac{5\cdot23}{13\cdot9}\cdot\text{etc.}=\frac{3}{2}\int\frac{dx}{\sqrt{(1-x^3)}},$$

$$\int\frac{dx}{\sqrt[3]{(1-x^3)^2}}=\frac{3}{1}\cdot\frac{2\cdot6}{4\cdot4}\cdot\frac{3\cdot15}{7\cdot7}\cdot\frac{4\cdot24}{10\cdot10}\cdot\frac{5\cdot33}{13\cdot13}\cdot\text{etc.}=\int\frac{dx}{\sqrt[3]{(1-x^3)^2}}$$

sive

$$=\frac{3}{1}\cdot\frac{2\cdot6}{4\cdot4}\cdot\frac{5\cdot9}{7.7}\cdot\frac{8\cdot12}{10\cdot10}\cdot\frac{11\cdot15}{13\cdot13}\cdot\text{etc.},$$

$$\int\frac{xdx}{\sqrt[3]{(1-x^3)^2}}=\frac{3}{2}\cdot\frac{2\cdot9}{4\cdot5}\cdot\frac{3\cdot18}{7\cdot8}\cdot\frac{4\cdot27}{10\cdot11}\cdot\frac{5\cdot36}{13\cdot14}\cdot\text{etc.}=\int\frac{dx}{\sqrt[3]{(1-x^3)}}$$

sive

$$=\frac{3}{2}\cdot\frac{3\cdot6}{4\cdot5}\cdot\frac{6\cdot9}{7\cdot8}\cdot\frac{9\cdot12}{10\cdot11}\cdot\frac{12\cdot15}{13\cdot14}\cdot\text{etc.},$$

$$\int\frac{dx}{\sqrt[3]{(1-x^4)^2}}=\frac{3}{1}\cdot\frac{2\cdot7}{4\cdot5}\cdot\frac{3\cdot19}{7\cdot9}\cdot\frac{4\cdot31}{10\cdot13}\cdot\frac{5\cdot43}{13\cdot17}\cdot\text{etc.}=\frac{3}{4}\int\frac{dx}{\sqrt[4]{(1-x^3)^3}},$$

$$\int\frac{xxdx}{\sqrt[3]{(1-x^4)^2}}=1\cdot\frac{2\cdot13}{4\cdot7}\cdot\frac{3\cdot25}{7\cdot11}\cdot\frac{4\cdot37}{10\cdot15}\cdot\frac{5\cdot49}{13\cdot19}\cdot\text{etc.}=\frac{3}{4}\int\frac{dx}{\sqrt[4]{(1-x^3)}}.$$

EXEMPLUM 3

372. Sit $\mu=2$ et $\nu=3$ fietque

$$\int\frac{x^{m-1}dx}{\sqrt[3]{(1-x^n)}}=\frac{3}{2m}\cdot\frac{2(3m+2n)}{5(m+n)}\cdot\frac{3(3m+5n)}{8(m+2n)}\cdot\frac{4(3m+8n)}{11(m+3n)}\cdot\text{etc.}=\frac{3}{n}\int\frac{xdx}{\sqrt[n]{(1-x^3)^{n-m}}},$$

unde sequentes casus speciales deducuntur

$$\int\frac{dx}{\sqrt[3]{(1-x^2)}}=\frac{3}{2}\cdot\frac{2\cdot7}{5\cdot3}\cdot\frac{3\cdot13}{8\cdot5}\cdot\frac{4\cdot19}{11\cdot7}\cdot\frac{5\cdot25}{14\cdot9}\cdot\text{etc.}=\frac{3}{2}\int\frac{xdx}{\sqrt{(1-x^3)}},$$

$$\int\frac{dx}{\sqrt[3]{(1-x^3)}}=\frac{3}{2}\cdot\frac{2\cdot9}{5\cdot4}\cdot\frac{3\cdot18}{8\cdot7}\cdot\frac{4\cdot27}{11\cdot10}\cdot\frac{5\cdot36}{14\cdot13}\cdot\text{etc.}=\int\frac{xdx}{\sqrt[3]{(1-x^3)^2}}$$

sive

$$=\frac{3\cdot3}{2\cdot4}\cdot\frac{6\cdot6}{5\cdot7}\cdot\frac{9\cdot9}{8\cdot10}\cdot\frac{12\cdot12}{11\cdot13}\cdot\text{etc.},$$

$$\int\frac{xdx}{\sqrt[3]{(1-x^3)}}=\frac{3}{4}\cdot\frac{2\cdot12}{5\cdot5}\cdot\frac{3\cdot21}{8\cdot8}\cdot\frac{4\cdot30}{11\cdot11}\cdot\frac{5\cdot39}{14\cdot14}\cdot\text{etc.}=\int\frac{xdx}{\sqrt[3]{(1-x^3)}}$$

sive

$$=\frac{3}{4}\cdot\frac{4\cdot6}{5\cdot5}\cdot\frac{7\cdot9}{8\cdot8}\cdot\frac{10\cdot12}{11\cdot11}\cdot\frac{13\cdot15}{14\cdot14}\cdot\text{etc.},$$

$$\int\frac{dx}{\sqrt[3]{(1-x^4)}}=\frac{3}{2}\cdot\frac{2\cdot11}{5\cdot5}\cdot\frac{3\cdot23}{8\cdot9}\cdot\frac{4\cdot35}{11\cdot13}\cdot\frac{5\cdot47}{14\cdot17}\cdot\text{etc.}=\frac{3}{4}\int\frac{xdx}{\sqrt[4]{(1-x^3)^3}},$$

$$\int\frac{x^2dx}{\sqrt[3]{(1-x^4)}}=\frac{1}{2}\cdot\frac{2\cdot17}{5\cdot7}\cdot\frac{3\cdot29}{8\cdot11}\cdot\frac{4\cdot41}{11\cdot15}\cdot\frac{5\cdot53}{14\cdot19}\cdot\text{etc.}=\frac{3}{4}\int\frac{xdx}{\sqrt[4]{(1-x^3)}}.$$

EXEMPLUM 4

373. Sit $\mu = 1$ et $\nu = 4$ fietque

$$\int \frac{x^{m-1}dx}{\sqrt[4]{(1-x^n)^3}} = \frac{4}{m} \cdot \frac{2(4m+n)}{5(m+n)} \cdot \frac{3(4m+5n)}{9(m+2n)} \cdot \frac{4(4m+9n)}{13(m+3n)} \cdot \text{etc.} = \frac{4}{n} \int \frac{dx}{\sqrt[n]{(1-x^4)^{n-m}}},$$

unde sequentes casus speciales prodeunt

$$\int \frac{dx}{\sqrt[4]{(1-x^2)^3}} = \frac{4}{1} \cdot \frac{2\cdot 6}{5\cdot 3} \cdot \frac{3\cdot 14}{9\cdot 5} \cdot \frac{4\cdot 22}{13\cdot 7} \cdot \frac{5\cdot 30}{17\cdot 9} \cdot \text{etc.} = 2\int \frac{dx}{\sqrt{(1-x^4)}}$$

seu

$$= \frac{4}{1} \cdot \frac{4\cdot 3}{3\cdot 5} \cdot \frac{6\cdot 7}{5\cdot 9} \cdot \frac{8\cdot 11}{7\cdot 13} \cdot \frac{10\cdot 15}{9\cdot 17} \cdot \text{etc.},$$

$$\int \frac{dx}{\sqrt[4]{(1-x^3)^3}} = \frac{4}{1} \cdot \frac{2\cdot 7}{5\cdot 4} \cdot \frac{3\cdot 19}{9\cdot 7} \cdot \frac{4\cdot 31}{13\cdot 10} \cdot \frac{5\cdot 43}{17\cdot 13} \cdot \text{etc.} = \frac{4}{3}\int \frac{dx}{\sqrt[3]{(1-x^4)^2}},$$

$$\int \frac{xdx}{\sqrt[4]{(1-x^3)^3}} = \frac{2}{1} \cdot \frac{2\cdot 11}{5\cdot 5} \cdot \frac{3\cdot 23}{9\cdot 8} \cdot \frac{4\cdot 35}{13\cdot 11} \cdot \frac{5\cdot 47}{17\cdot 14} \cdot \text{etc.} = \frac{4}{3}\int \frac{dx}{\sqrt[3]{(1-x^4)}},$$

$$\int \frac{dx}{\sqrt[4]{(1-x^4)^3}} = \frac{4}{1} \cdot \frac{2\cdot 8}{5\cdot 5} \cdot \frac{3\cdot 24}{9\cdot 9} \cdot \frac{4\cdot 40}{13\cdot 13} \cdot \frac{5\cdot 56}{17\cdot 17} \cdot \text{etc.} = \int \frac{dx}{\sqrt[4]{(1-x^4)^3}}$$

seu

$$= \frac{4}{1} \cdot \frac{4\cdot 4}{5\cdot 5} \cdot \frac{6\cdot 12}{9\cdot 9} \cdot \frac{8\cdot 20}{13\cdot 13} \cdot \frac{10\cdot 28}{17\cdot 17} \cdot \text{etc.}$$

seu

$$= \frac{4}{1} \cdot \frac{2\cdot 8}{5\cdot 5} \cdot \frac{6\cdot 12}{9\cdot 9} \cdot \frac{10\cdot 16}{13\cdot 13} \cdot \frac{14\cdot 20}{17\cdot 17} \cdot \text{etc.},$$

$$\int \frac{xxdx}{\sqrt[4]{(1-x^4)^3}} = \frac{4}{3} \cdot \frac{2\cdot 16}{5\cdot 7} \cdot \frac{3\cdot 32}{9\cdot 11} \cdot \frac{4\cdot 48}{13\cdot 15} \cdot \frac{5\cdot 64}{17\cdot 19} \cdot \text{etc.} = \int \frac{dx}{\sqrt[4]{(1-x^4)}}$$

seu

$$= \frac{4}{3} \cdot \frac{4\cdot 8}{5\cdot 7} \cdot \frac{6\cdot 16}{9\cdot 11} \cdot \frac{8\cdot 24}{13\cdot 15} \cdot \frac{10\cdot 32}{17\cdot 19} \cdot \text{etc.}$$

seu

$$= \frac{4}{3} \cdot \frac{4\cdot 8}{5\cdot 7} \cdot \frac{8\cdot 12}{9\cdot 11} \cdot \frac{12\cdot 16}{13\cdot 15} \cdot \frac{16\cdot 20}{17\cdot 19} \cdot \text{etc.}$$

Atque in his et praecedentibus iam casus $\mu = 3$ et $\nu = 4$ est contentus.

SCHOLION

374. Caeterum hae formulae, in quas litteras μ et ν introduxi, latius non patent quam primum consideratae; series enim pendent a binis fractionibus $\frac{m}{n}$ et $\frac{\mu}{\nu}$, quae cum semper ad communem denominatorem revocari queant, formulas

$$\int\frac{x^{m-1}dx}{\sqrt[n]{(1-x^n)^{n-k}}}=\int\frac{x^{k-1}dx}{\sqrt[n]{(1-x^n)^{n-m}}}$$

perpendisse sufficiet. Cum igitur earum valor casu $x=1$ aequetur huic producto

$$\frac{1}{k}\cdot\frac{n(m+k)}{m(k+n)}\cdot\frac{2n(m+k+n)}{(m+n)(k+2n)}\cdot\frac{3n(m+k+2n)}{(m+2n)(k+3n)}\cdot\text{etc.},$$

si in singulis membris factores numeratorum permutemus et membra aliter partiamur, idem productum hanc induet formam

$$\frac{m+k}{mk}\cdot\frac{n(m+k+n)}{(m+n)(k+n)}\cdot\frac{2n(m+k+2n)}{(m+2n)(k+2n)}\cdot\frac{3n(m+k+3n)}{(m+3n)(k+3n)}\cdot\text{etc.},$$

quae ad memoriam magis accommodata videtur. Simili modo cum sit

$$\int\frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}=\int\frac{x^{q-1}dx}{\sqrt[n]{(1-x^n)^{n-p}}}$$

$$=\frac{p+q}{pq}\cdot\frac{n(p+q+n)}{(p+n)(q+n)}\cdot\frac{2n(p+q+2n)}{(p+2n)(q+2n)}\cdot\frac{3n(p+q+3n)}{(p+3n)(q+3n)}\cdot\text{etc.},$$

illam formam per hanc dividendo erit

$$\frac{\int x^{m-1}dx(1-x^n)^{\frac{k-n}{n}}}{\int x^{p-1}dx(1-x^n)^{\frac{q-n}{n}}}$$

$$=\frac{pq(m+k)}{mk(p+q)}\cdot\frac{(p+n)(q+n)(m+k+n)}{(m+n)(k+n)(p+q+n)}\cdot\frac{(p+2n)(q+2n)(m+k+2n)}{(m+2n)(k+2n)(p+q+2n)}\cdot\text{etc.},$$

cuius omnia membra eadem lege continentur. Hinc autem eximiae comparationes huiusmodi formularum deduci possunt, quae quo facilius commemorari queant, brevitatis causa sequenti scriptionis compendio utar.

DEFINITIO

375. *Formulae integralis*

$$\int x^{p-1}dx(1-x^n)^{\frac{q-n}{n}}$$

valorem, quem posito $x=1$ *recipit, brevitatis gratia hoc signo* $\left(\frac{p}{q}\right)$ *indicemus, ubi quidem exponentem* n, *quem in comparatione plurium huiusmodi formularum eundem esse assumo, subintelligi oportet.*

COROLLARIUM 1

376. Primum igitur patet esse $\left(\frac{p}{q}\right)=\left(\frac{q}{p}\right)$ et utramque formulam esse

$$=\frac{p+q}{pq}\cdot\frac{n(p+q+n)}{(p+n)(q+n)}\cdot\frac{2n(p+q+2n)}{(p+2n)(q+2n)}\cdot\text{etc.},$$

quorum membrorum progressio est manifesta, dum singuli factores tam numeratoris quam denominatoris continuo eodem numero n augentur, ita ut ex cognito primo membro sequentia facile formentur.

COROLLARIUM 2

377. Deinde si sit $p=n$, ob formulam integrabilem liquet esse

$$\left(\frac{n}{q}\right)=\left(\frac{q}{n}\right)=\frac{1}{q},\quad\text{item}\quad\left(\frac{p}{n}\right)=\left(\frac{n}{p}\right)=\frac{1}{p}$$

[pro $q=n$]. Porro cum

$$\int x^{p-1}dx(1-x^n)^{-\frac{p}{n}}=\frac{\pi}{n\sin.\frac{p\pi}{n}},$$

ob $q-n=-p$ seu $p+q=n$ erit

$$\left(\frac{p}{n-p}\right)=\left(\frac{n-p}{p}\right)=\frac{\pi}{n\sin.\frac{p\pi}{n}}.$$

Quare valor formulae $\left(\frac{p}{q}\right)$ absolute assignari potest, quoties fuerit vel $p=n$ vel $q=n$ vel $p+q=n$.

COROLLARIUM 3

378. Quia etiam [§ 345] invenimus hanc reductionem

$$\int x^{p+n-1}dx(1-x^n)^{\frac{q-n}{n}} = \frac{p}{p+q}\int x^{p-1}dx(1-x^n)^{\frac{q-n}{n}},$$

sequitur fore

$$\left(\frac{p+n}{q}\right) = \frac{p}{p+q}\left(\frac{p}{q}\right)$$

hincque

$$\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right) = \frac{p-n}{p+q-n}\left(\frac{p-n}{q}\right) = \frac{q-n}{p+q-n}\left(\frac{p}{q-n}\right),$$

tum vero etiam

$$\left(\frac{p}{q}\right) = \frac{(p-n)(q-n)}{(p+q-n)(p+q-2n)}\cdot\left(\frac{p-n}{q-n}\right),$$

unde semper numeri p et q infra n deprimi possunt.

PROBLEMA 46

379. *Invenire diversa producta ex binis huiusmodi formulis, quae inter se sint aequalia.*

SOLUTIO

Quaerantur ergo numeri a, b, c, d et p, q, r, s, ut fiat

$$\left(\frac{a}{b}\right)\left(\frac{c}{d}\right) = \left(\frac{p}{q}\right)\left(\frac{r}{s}\right),$$

quod, cum sit

$$\left(\frac{a}{b}\right) = \frac{a+b}{ab}\cdot\frac{n(a+b+n)}{(a+n)(b+n)}\cdot\text{etc.},\quad \left(\frac{c}{d}\right) = \frac{c+d}{cd}\cdot\frac{n(c+d+n)}{(c+n)(d+n)}\cdot\text{etc.},$$

$$\left(\frac{p}{q}\right) = \frac{p+q}{pq}\cdot\frac{n(p+q+n)}{(p+n)(q+n)}\cdot\text{etc.},\quad \left(\frac{r}{s}\right) = \frac{r+s}{rs}\cdot\frac{n(r+s+n)}{(r+n)(s+n)}\cdot\text{etc.},$$

eveniet, si fuerit

$$\frac{(a+b)(c+d)}{abcd} = \frac{(p+q)(r+s)}{pqrs}$$

seu

$$abcd(p+q)(r+s) = pqrs(a+b)(c+d),$$

ita ut, cum utrinque sex sint factores, singuli singulis sint aequales. Ex quaternis ergo $abcd$ et $pqrs$ binos ad minimum aequales esse oportet; sit itaque $s = d$ efficique oportet

$$abc(p+q)(r+d) = pqr(a+b)(c+d).$$

I. Sumatur alter factor r; qui cum ipsi c aequari nequeat, quia alioquin fieret $\left(\frac{c}{d}\right) = \left(\frac{r}{s}\right)$, statuatur $r = b$, ut fiat

$$ac(p+q)(b+d) = pq(a+b)(c+d);$$

hic neque p neque q ipsi $p+q$ aequari potest, poni ergo debet:

1) Vel $p+q = a+b$, ut sit $ac(b+d) = pq(c+d)$, quia neque c neque $b+d$ ipsi $c+d$ aequari potest; fieret enim vel $d=0$ vel $b=c$ et $\left(\frac{r}{s}\right) = \left(\frac{c}{d}\right)$; relinquitur $a = c+d$ et $pq = c(b+d)$ ideoque $p = b+d$ et $q = c$, unde conficitur

$$\left(\frac{c+d}{b}\right)\left(\frac{c}{d}\right) = \left(\frac{b+d}{c}\right)\left(\frac{b}{d}\right).$$

2) Vel $p+q = c+d$, ergo $ac(b+d) = pq(a+b)$; hic c neque ipsi p neque q aequari potest; fieret enim $\left(\frac{p}{q}\right) = \left(\frac{c}{d}\right)$; unde fiat $c = a+b$, ut sit $pq = a(b+d)$, ergo $p = a$, $q = b+d$, $r = b$, $s = d$, consequenter

$$\left(\frac{a}{b}\right)\left(\frac{a+b}{d}\right) = \left(\frac{b+d}{a}\right)\left(\frac{b}{d}\right).$$

II. Quia $r = a$ non differt a praecedenti ob a et b permutabiles, statuatur $r = p+q$ fietque $abc(d+p+q) = pq(a+b)(c+d)$. Quoniam r ipsi c aequari nequit, factor $d+p+q$ neque ipsi p neque q neque $c+d$ aequalis poni potest; relinquitur ergo $d+p+q = a+b$ et $abc = pq(c+d)$; ubi, quia c ipsi $c+d$ aequari nequit ac p et q pari conditione gaudent, fiat $p = c$; erit $q = a+b-c-d$ et $ab = (c+d)(a+b-c-d)$, unde $a = c+d$, $q = b$, $p = c$, $r = b+c$, $s = d$, sicque conficitur

$$\left(\frac{c+d}{b}\right)\left(\frac{c}{d}\right) = \left(\frac{c}{b}\right)\left(\frac{b+c}{d}\right).$$

COROLLARIUM 1

380. Hae solutiones eodem fere redeunt indeque tria producta binarum formularum aequalia eruuntur

$$\left(\frac{c}{d}\right)\left(\frac{c+d}{b}\right)=\left(\frac{c}{b}\right)\left(\frac{b+c}{d}\right)=\left(\frac{b}{d}\right)\left(\frac{b+d}{c}\right)$$

vel in litteris p, q, r

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)=\left(\frac{q}{r}\right)\left(\frac{q+r}{p}\right)=\left(\frac{p}{r}\right)\left(\frac{p+r}{q}\right).$$

COROLLARIUM 2

381. Si hae formulae in producta infinita evolvantur, reperietur

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)=\frac{p+q+r}{pqr}\cdot\frac{nn(p+q+r+n)}{(p+n)(q+n)(r+n)}\cdot\frac{4nn(p+q+r+2n)}{(p+2n)(q+2n)(r+2n)}\cdot\text{etc.},$$

unde patet tres litteras p, q, r utcunque inter se permutari posse, atque hinc ternas illas formulas concludere licet.

COROLLARIUM 3

382. Restituamus ipsas formulas integrales et sequentia tria producta erunt inter se aequalia

$$\int\frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}\cdot\int\frac{x^{p+q-1}dx}{\sqrt[n]{(1-x^n)^{n-r}}}=\int\frac{x^{q-1}dx}{\sqrt[n]{(1-x^n)^{n-r}}}\cdot\int\frac{x^{q+r-1}dx}{\sqrt[n]{(1-x^n)^{n-p}}}$$
$$=\int\frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-r}}}\cdot\int\frac{x^{p+r-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}.$$

COROLLARIUM 4

383. Hic casus notatu dignus, quo $p+q=n$; tum enim ob

$$\left(\frac{p+q}{r}\right)=\left(\frac{n}{r}\right)=\frac{1}{r}\quad\text{et}\quad\left(\frac{p}{q}\right)=\frac{\pi}{n\sin.\frac{p\pi}{n}}$$

haec tria producta fient $=\dfrac{\pi}{nr\sin.\frac{p\pi}{n}}$. Erit scilicet

$$\int\frac{x^{n-p-1}dx}{\sqrt[n]{(1-x^n)^{n-r}}}\cdot\int\frac{x^{n-p+r-1}dx}{\sqrt[n]{(1-x^n)^{n-p}}}=\int\frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-r}}}\cdot\int\frac{x^{p+r-1}dx}{\sqrt[n]{(1-x^n)^{p}}}=\frac{\pi}{nr\sin.\frac{p\pi}{n}}.$$

SCHOLION

384. Triplex ista proprietas productorum ex binis formulis maxime est notatu digna ac pro variis numeris loco p, q, r substituendis obtinebuntur sequentes aequalitates speciales

p	q	r	
1	1	2	$\left(\frac{1}{1}\right)\left(\frac{2}{2}\right)=\left(\frac{2}{1}\right)\left(\frac{3}{1}\right)$
1	2	2	$\left(\frac{2}{1}\right)\left(\frac{3}{2}\right)=\left(\frac{2}{2}\right)\left(\frac{4}{1}\right)$
1	2	3	$\left(\frac{2}{1}\right)\left(\frac{3}{3}\right)=\left(\frac{3}{2}\right)\left(\frac{5}{1}\right)=\left(\frac{3}{1}\right)\left(\frac{4}{2}\right)$
1	1	3	$\left(\frac{1}{1}\right)\left(\frac{3}{2}\right)=\left(\frac{3}{1}\right)\left(\frac{4}{1}\right)$
2	2	3	$\left(\frac{2}{2}\right)\left(\frac{4}{3}\right)=\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)$
1	3	3	$\left(\frac{3}{1}\right)\left(\frac{4}{3}\right)=\left(\frac{3}{3}\right)\left(\frac{6}{1}\right)$
2	3	3	$\left(\frac{3}{2}\right)\left(\frac{5}{3}\right)=\left(\frac{3}{3}\right)\left(\frac{6}{2}\right)$
1	1	4	$\left(\frac{1}{1}\right)\left(\frac{4}{2}\right)=\left(\frac{4}{1}\right)\left(\frac{5}{1}\right)$
1	2	4	$\left(\frac{2}{1}\right)\left(\frac{4}{3}\right)=\left(\frac{4}{2}\right)\left(\frac{6}{1}\right)=\left(\frac{4}{1}\right)\left(\frac{5}{2}\right)$
1	3	4	$\left(\frac{3}{1}\right)\left(\frac{4}{4}\right)=\left(\frac{4}{1}\right)\left(\frac{5}{3}\right)=\left(\frac{4}{3}\right)\left(\frac{7}{1}\right)$
1	4	4	$\left(\frac{4}{1}\right)\left(\frac{5}{4}\right)=\left(\frac{4}{4}\right)\left(\frac{8}{1}\right)$
2	2	4	$\left(\frac{2}{2}\right)\left(\frac{4}{4}\right)=\left(\frac{4}{2}\right)\left(\frac{6}{2}\right)$
2	3	4	$\left(\frac{3}{2}\right)\left(\frac{5}{4}\right)=\left(\frac{4}{3}\right)\left(\frac{7}{2}\right)=\left(\frac{4}{2}\right)\left(\frac{6}{3}\right)$
2	4	4	$\left(\frac{4}{2}\right)\left(\frac{6}{4}\right)=\left(\frac{4}{4}\right)\left(\frac{8}{2}\right)$
3	3	4	$\left(\frac{3}{3}\right)\left(\frac{6}{4}\right)=\left(\frac{4}{3}\right)\left(\frac{7}{3}\right)$
3	4	4	$\left(\frac{4}{3}\right)\left(\frac{7}{4}\right)=\left(\frac{4}{4}\right)\left(\frac{8}{3}\right).$

Quae formulae pro omnibus numeris n valent, ac si numeri maiores quam n occurrant, eos ad minores reduci posse supra vidimus.

PROBLEMA 47

385. *Invenire producta diversa ex ternis huiusmodi formulis, quae inter se sint aequalia.*

SOLUTIO

Consideretur productum $\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+q+r}{s}\right)$, quod evolutum praebet

$$\frac{p+q+r+s}{pqrs}\cdot\frac{n^3(p+q+r+s+n)}{(p+n)(q+n)(r+n)(s+n)}\cdot\text{etc.},$$

quod eundem valorem retinere evidens est, quomodocunque quatuor litterae inter se commutentur. Tum vero eadem evolutio prodit ex hoc producto $\left(\frac{p}{q}\right)\left(\frac{r}{s}\right)\left(\frac{p+q}{r+s}\right)$, ubi eadem permutatio locum habet. Aequalia ergo sunt inter se omnia haec producta

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+q+r}{s}\right),\quad \left(\frac{p}{r}\right)\left(\frac{p+r}{q}\right)\left(\frac{p+q+r}{s}\right),\quad \left(\frac{p}{s}\right)\left(\frac{p+s}{q}\right)\left(\frac{p+q+s}{r}\right),$$

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{s}\right)\left(\frac{p+q+s}{r}\right),\quad \left(\frac{p}{r}\right)\left(\frac{p+r}{s}\right)\left(\frac{p+r+s}{q}\right),\quad \left(\frac{p}{s}\right)\left(\frac{p+s}{r}\right)\left(\frac{p+r+s}{q}\right),$$

$$\left(\frac{q}{r}\right)\left(\frac{q+r}{p}\right)\left(\frac{p+q+r}{s}\right),\quad \left(\frac{q}{s}\right)\left(\frac{q+s}{p}\right)\left(\frac{p+q+s}{r}\right),\quad \left(\frac{r}{s}\right)\left(\frac{r+s}{p}\right)\left(\frac{p+r+s}{q}\right),$$

$$\left(\frac{q}{r}\right)\left(\frac{q+r}{s}\right)\left(\frac{q+r+s}{p}\right),\quad \left(\frac{q}{s}\right)\left(\frac{q+s}{r}\right)\left(\frac{q+r+s}{p}\right),\quad \left(\frac{r}{s}\right)\left(\frac{r+s}{q}\right)\left(\frac{q+r+s}{p}\right).$$

Producta alterius formae ope praecedentis proprietatis hinc sponte fluunt; est enim

$$\left(\frac{p+q}{r}\right)\left(\frac{p+q+r}{s}\right)=\left(\frac{r}{s}\right)\left(\frac{r+s}{p+q}\right).$$

Deinde vero etiam hoc productum $\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+r}{s}\right)$ evolutum pro primo membro dat $\frac{(p+q+r)(p+r+s)}{pqrs(p+r)}$, in quo tam p et r quam q et s inter se permutare licet, ita ut sit

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+r}{s}\right)=\left(\frac{r}{s}\right)\left(\frac{r+s}{p}\right)\left(\frac{p+r}{q}\right).$$

SCHOLION

386. Quantumvis late haec patere videantur, tamen nullas novas comparationes suppeditant, quae non iam in praecedenti contineantur. Postrema enim aequalitas

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+r}{s}\right)=\left(\frac{r}{s}\right)\left(\frac{r+s}{p}\right)\left(\frac{p+r}{q}\right)$$

oritur ex multiplicatione harum

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)=\left(\frac{p}{r}\right)\left(\frac{p+r}{q}\right),\qquad \left(\frac{p}{r}\right)\left(\frac{p+r}{s}\right)=\left(\frac{r}{s}\right)\left(\frac{r+s}{p}\right).$$

Priorum vero formatio ex hoc exemplo patebit: aequalitas

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+q+r}{s}\right)=\left(\frac{r}{s}\right)\left(\frac{r+s}{p}\right)\left(\frac{p+r+s}{q}\right)$$

oritur ex multiplicatione harum

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r+s}\right)=\left(\frac{r+s}{p}\right)\left(\frac{p+r+s}{q}\right),\qquad \left(\frac{p+q}{r}\right)\left(\frac{p+q+r}{s}\right)=\left(\frac{r}{s}\right)\left(\frac{r+s}{p+q}\right).$$

Istae autem comparationes praecipue utiles sunt ad valores diversarum formularum eiusdem ordinis seu pro dato numero n invicem reducendos, ut integratio ad paucissimas revocetur, quibus datis reliquae per eas definiri queant.

PROBLEMA 48

387. *Formulas simplicissimas exhibere, ad quas integratio omnium casuum in forma*

$$\left(\frac{p}{q}\right)=\int\frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}$$

contentorum reduci queat.

SOLUTIO

Primo est $\left(\frac{n}{p}\right)=\frac{1}{p}$, unde habentur hi casus

$$\left(\frac{n}{1}\right)=1,\quad \left(\frac{n}{2}\right)=\frac{1}{2},\quad \left(\frac{n}{3}\right)=\frac{1}{3},\quad \left(\frac{n}{4}\right)=\frac{1}{4},\quad \left(\frac{n}{5}\right)=\frac{1}{5}\quad \text{etc.}$$

Deinde est $\left(\frac{p}{n-p}\right)=\frac{\pi}{n\sin.\frac{p\pi}{n}}$, unde omnium harum formularum valores sunt cogniti, quas indicemus

$$\left(\frac{n-1}{1}\right)=\alpha,\quad \left(\frac{n-2}{2}\right)=\beta,\quad \left(\frac{n-3}{3}\right)=\gamma,\quad \left(\frac{n-4}{4}\right)=\delta \quad \text{etc.}$$

Verum hi non sufficiunt ad reliquos omnes expediendos, praeterea tanquam cognitos spectari oportet hos

$$\left(\frac{n-2}{1}\right)=A,\quad \left(\frac{n-3}{2}\right)=B,\quad \left(\frac{n-4}{3}\right)=C,\quad \left(\frac{n-5}{4}\right)=D \quad \text{etc.}$$

atque ex his reliqui omnes determinari poterunt ope aequationum supra demonstratarum; unde potissimum has notasse iuvabit

$$\left(\frac{n-a}{a}\right)\left(\frac{n}{b}\right)=\left(\frac{n-a}{b}\right)\left(\frac{n-a+b}{a}\right),$$

$$\left(\frac{n-a}{a}\right)\left(\frac{n-a-b}{b}\right)=\left(\frac{n-b}{b}\right)\left(\frac{n-a-b}{a}\right),$$

$$\left(\frac{n-a}{a}\right)\left(\frac{n-b-1}{b}\right)\left(\frac{n-a-b}{a-1}\right)=\left(\frac{n-b}{b}\right)\left(\frac{n-a}{a-1}\right)\left(\frac{n-a-b}{a}\right).$$

Ex harum prima posito $a=b+1$ invenitur

$$\left(\frac{n-1}{a}\right)=\left(\frac{n-a}{a}\right)\left(\frac{n}{a-1}\right):\left(\frac{n-a}{a-1}\right),$$

ubi $\left(\frac{n}{a-1}\right)=\frac{1}{a-1}$, ideoque per formulas assumtas definitur $\left(\frac{n-1}{a}\right)$.

Ex secunda posito $b=1$ invenitur

$$\left(\frac{n-a-1}{1}\right)=\left(\frac{n-1}{1}\right)\left(\frac{n-a-1}{a}\right):\left(\frac{n-a}{a}\right).$$

Ex tertia posito $b=1$ deducitur

$$\left(\frac{n-a-1}{a-1}\right)=\left(\frac{n-1}{1}\right)\left(\frac{n-a}{a-1}\right)\left(\frac{n-a-1}{a}\right):\left(\frac{n-a}{a}\right)\left(\frac{n-2}{1}\right)$$

sicque reperiuntur omnes formulae $\left(\frac{n-a-2}{a}\right)$ et ex his porro ponendo $b=2$ in tertia

$$\left(\frac{n-a-2}{a-1}\right)=\left(\frac{n-2}{2}\right)\left(\frac{n-a}{a-1}\right)\left(\frac{n-a-2}{a}\right):\left(\frac{n-a}{a}\right)\left(\frac{n-3}{2}\right),$$

unde reperiuntur formae $\left(\frac{n-a-3}{a}\right)$ et ita porro omnes $\left(\frac{n-a-b}{a}\right)$, quippe quae forma omnes complectitur. Labor autem per priores aequationes non mediocriter contrahitur. Inventa enim $\left(\frac{n-a-2}{a}\right)$ ex prima colligitur

$$\left(\frac{n-2}{a+2}\right)=\left(\frac{n-a-2}{a+2}\right)\left(\frac{n}{a}\right):\left(\frac{n-a-2}{a}\right),$$

ex secunda vero

$$\left(\frac{n-a-2}{2}\right)=\left(\frac{n-2}{2}\right)\left(\frac{n-a-2}{a}\right):\left(\frac{n-a}{a}\right)$$

similique modo ex inventis formulis $\left(\frac{n-a-3}{a}\right)$ derivantur hae

$$\left(\frac{n-3}{a+3}\right)=\left(\frac{n-a-3}{a+3}\right)\left(\frac{n}{a}\right):\left(\frac{n-a-3}{a}\right),$$

$$\left(\frac{n-a-3}{3}\right)=\left(\frac{n-3}{3}\right)\left(\frac{n-a-3}{a}\right):\left(\frac{n-a}{a}\right).$$

COROLLARIUM 1

388. Ex aequatione $\left(\frac{n-1}{a}\right)=\frac{1}{a-1}\left(\frac{n-a}{a}\right):\left(\frac{n-a}{a-1}\right)$ definiuntur

$$\left(\frac{n-1}{2}\right)=\frac{\beta}{1A},\quad \left(\frac{n-1}{3}\right)=\frac{\gamma}{2B},\quad \left(\frac{n-1}{4}\right)=\frac{\delta}{3C},\quad \left(\frac{n-1}{5}\right)=\frac{\varepsilon}{4D}\quad \text{etc.},$$

ex aequatione vero $\left(\frac{n-a-1}{1}\right)=\left(\frac{n-1}{1}\right)\left(\frac{n-a-1}{a}\right):\left(\frac{n-a}{a}\right)$ hae formulae

$$\left(\frac{n-2}{1}\right)=\frac{\alpha A}{\alpha},\quad \left(\frac{n-3}{1}\right)=\frac{\alpha B}{\beta},\quad \left(\frac{n-4}{1}\right)=\frac{\alpha C}{\gamma},\quad \left(\frac{n-5}{1}\right)=\frac{\alpha D}{\delta}\quad \text{etc.}$$

COROLLARIUM 2

389. Aequatio $\left(\frac{n-a-1}{a-1}\right)=\left(\frac{n-1}{1}\right)\left(\frac{n-a}{a-1}\right)\left(\frac{n-a-1}{a}\right):\left(\frac{n-a}{a}\right)\left(\frac{n-2}{1}\right)$ praebet

$$\left(\frac{n-3}{1}\right)=\frac{\alpha AB}{\beta A},\quad \left(\frac{n-4}{2}\right)=\frac{\alpha BC}{\gamma A},\quad \left(\frac{n-5}{3}\right)=\frac{\alpha CD}{\delta A},\quad \left(\frac{n-6}{4}\right)=\frac{\alpha DE}{\varepsilon A}\quad \text{etc.},$$

unde reperiuntur istae formulae $\left(\frac{n-2}{a+2}\right)=\left(\frac{n-a-2}{a+2}\right)\left(\frac{n}{a}\right):\left(\frac{n-a-2}{a}\right)$ [, quae sunt]

$$\left(\frac{n-2}{3}\right)=\frac{\gamma\beta A}{1\alpha AB},\quad \left(\frac{n-2}{4}\right)=\frac{\delta\gamma A}{2\alpha BC},\quad \left(\frac{n-2}{5}\right)=\frac{\varepsilon\delta A}{3\alpha CD},\quad \left(\frac{n-2}{6}\right)=\frac{\zeta\varepsilon A}{4\alpha DE}\quad \text{etc.}$$

atque etiam istae $\left(\frac{n-a-2}{2}\right)=\left(\frac{n-2}{2}\right)\left(\frac{n-a-2}{a}\right):\left(\frac{n-a}{a}\right)$, quae sunt

$$\left(\frac{n-3}{2}\right)=\frac{\beta\alpha AB}{\alpha\beta A},\quad \left(\frac{n-4}{2}\right)=\frac{\beta\alpha BC}{\beta\gamma A},\quad \left(\frac{n-5}{2}\right)=\frac{\beta\alpha CD}{\gamma\delta A},\quad \left(\frac{n-6}{2}\right)=\frac{\beta\alpha DE}{\delta\varepsilon A}\quad \text{etc.}$$

COROLLARIUM 3

390. Tum aequatio $\left(\frac{n-a-2}{a-1}\right)=\left(\frac{n-2}{2}\right)\left(\frac{n-a}{a-1}\right)\left(\frac{n-a-2}{a}\right):\left(\frac{n-a}{a}\right)\left(\frac{n-3}{2}\right)$ dat

$$\left(\frac{n-4}{1}\right)=\frac{\alpha\beta ABC}{\beta\gamma AB},\quad \left(\frac{n-5}{2}\right)=\frac{\alpha\beta BCD}{\gamma\delta AB},\quad \left(\frac{n-6}{3}\right)=\frac{\alpha\beta CDE}{\delta\varepsilon AB},\quad \left(\frac{n-7}{4}\right)=\frac{\alpha\beta DEF}{\varepsilon\zeta AB}\quad \text{etc.},$$

hinc $\left(\frac{n-3}{a+3}\right)=\left(\frac{n-a-3}{a+3}\right)\left(\frac{n}{a}\right):\left(\frac{n-a-3}{a}\right)$ praebet

$$\left(\frac{n-3}{4}\right)=\frac{\beta\gamma\delta AB}{1\alpha\beta ABC},\quad \left(\frac{n-3}{5}\right)=\frac{\gamma\delta\varepsilon AB}{2\alpha\beta BCD},\quad \left(\frac{n-3}{6}\right)=\frac{\delta\varepsilon\zeta AB}{3\alpha\beta CDE}\quad \text{etc.}$$

atque ex $\left(\frac{n-a-3}{3}\right)=\left(\frac{n-3}{3}\right)\left(\frac{n-a-3}{a}\right):\left(\frac{n-a}{a}\right)$ deducuntur

$$\left(\frac{n-5}{3}\right)=\frac{\alpha\beta\gamma BCD}{\beta\gamma\delta AB},\quad \left(\frac{n-6}{3}\right)=\frac{\alpha\beta\gamma CDE}{\gamma\delta\varepsilon AB},\quad \left(\frac{n-7}{3}\right)=\frac{\alpha\beta\gamma DEF}{\delta\varepsilon\zeta AB}\quad \text{etc.}$$

EXEMPLUM 1

391. *Casus in hac forma* $\int\frac{x^{p-1}dx}{\sqrt[2]{(1-x^2)^{2-q}}}=\left(\frac{p}{q}\right)$ *contentos, ubi* $n=2$, *evolvere, ubi est* $\left(\frac{p+2}{q}\right)=\frac{p}{p+q}\left(\frac{p}{q}\right)$.

Manifestum est has formulas omnes vel algebraice vel per angulos expediri; his tamen regulis utentes, quia numeri p et q binarium superare non debent, unam formulam a circulo pendentem habemus

$$\left(\frac{1}{1}\right) = \frac{\pi}{2\sin.\frac{\pi}{2}} = \frac{\pi}{2} = \alpha,$$

unde nostri casus erunt

$$\left(\frac{2}{1}\right) = 1, \quad \left(\frac{2}{2}\right) = \frac{1}{2},$$

$$\left(\frac{1}{1}\right) = \alpha.$$

EXEMPLUM 2

392. *Casus in hac forma* $\int \frac{x^{p-1}dx}{\sqrt[3]{(1-x^3)^{3-q}}} = \left(\frac{p}{q}\right)$ *contentos, ubi* $n=3$, *evolvere, ubi est* $\left(\frac{p+3}{q}\right) = \frac{p}{p+q}\left(\frac{p}{q}\right)$.

Hic casus principales, ad quos caeteri reducuntur, sunt

$$\left(\frac{2}{1}\right) = \frac{\pi}{3\sin.\frac{\pi}{3}} = \frac{2\pi}{3\sqrt{3}} = \alpha \quad \text{et} \quad \left(\frac{1}{1}\right) = A = \int \frac{dx}{\sqrt[3]{(1-x^3)^2}},$$

qua concessa erunt reliqui

$$\left(\frac{3}{1}\right) = 1, \quad \left(\frac{3}{2}\right) = \frac{1}{2}, \quad \left(\frac{3}{3}\right) = \frac{1}{3},$$

$$\left(\frac{2}{1}\right) = \alpha, \quad \left(\frac{2}{2}\right) = \frac{\alpha}{A},$$

$$\left(\frac{1}{1}\right) = A.$$

EXEMPLUM 3

393. *Casus in hac forma* $\int \frac{x^{p-1}dx}{\sqrt[4]{(1-x^4)^{4-q}}} = \left(\frac{p}{q}\right)$ *contentos, ubi* $n=4$, *evolvere, ubi est* $\left(\frac{p+4}{q}\right) = \frac{p}{p+q}\left(\frac{p}{q}\right)$.

A circulo pendent hae duae

$$\left(\frac{3}{1}\right) = \frac{\pi}{4\sin.\frac{\pi}{4}} = \frac{\pi}{2\sqrt{2}} = \alpha \quad \text{et} \quad \left(\frac{2}{2}\right) = \frac{\pi}{4\sin.\frac{2\pi}{4}} = \frac{\pi}{4} = \beta,$$

praeterea vero una transcendente singulari opus est $\left(\frac{2}{1}\right) = A$, unde reliquae ita determinantur

$$\left(\frac{4}{1}\right) = 1, \quad \left(\frac{4}{2}\right) = \frac{1}{2}, \quad \left(\frac{4}{3}\right) = \frac{1}{3}, \quad \left(\frac{4}{4}\right) = \frac{1}{4},$$

$$\left(\frac{3}{1}\right) = \alpha, \quad \left(\frac{3}{2}\right) = \frac{\beta}{A}, \quad \left(\frac{3}{3}\right) = \frac{\alpha}{2A},$$

$$\left(\frac{2}{1}\right) = A, \quad \left(\frac{2}{2}\right) = \beta,$$

$$\left(\frac{1}{1}\right) = \frac{\alpha A}{\beta}.$$

EXEMPLUM 4

394. *Casus in hac forma* $\int \frac{x^{p-1}dx}{\sqrt[5]{(1-x^5)^{5-q}}} = \left(\frac{p}{q}\right)$ *contentos, ubi* $n = 5$, *evolvere, ubi est* $\left(\frac{p+5}{q}\right) = \frac{p}{p+q}\left(\frac{p}{q}\right)$.

A circulo pendent hae duae formulae

$$\left(\frac{4}{1}\right) = \frac{\pi}{5\,\text{sin.}\,\frac{\pi}{5}} = \alpha \quad \text{et} \quad \left(\frac{3}{2}\right) = \frac{\pi}{5\,\text{sin.}\,\frac{2\pi}{5}} = \beta,$$

praeter quas duas novas transcendentes assumi oportet

$$\left(\frac{3}{1}\right) = A \quad \text{et} \quad \left(\frac{2}{2}\right) = B,$$

per quas omnes sequenti modo determinantur

$$\left(\frac{5}{1}\right) = 1, \quad \left(\frac{5}{2}\right) = \frac{1}{2}, \quad \left(\frac{5}{3}\right) = \frac{1}{3}, \quad \left(\frac{5}{4}\right) = \frac{1}{4}, \quad \left(\frac{5}{5}\right) = \frac{1}{5},$$

$$\left(\frac{4}{1}\right) = \alpha, \quad \left(\frac{4}{2}\right) = \frac{\beta}{A}, \quad \left(\frac{4}{3}\right) = \frac{\beta}{2B}, \quad \left(\frac{4}{4}\right) = \frac{\alpha}{3A},$$

$$\left(\frac{3}{1}\right) = A, \quad \left(\frac{3}{2}\right) = \beta, \quad \left(\frac{3}{3}\right) = \frac{\beta\beta}{\alpha B},$$

$$\left(\frac{2}{1}\right) = \frac{\alpha B}{\beta}, \quad \left(\frac{2}{2}\right) = B,$$

$$\left(\frac{1}{1}\right) = \frac{\alpha A}{\beta}.$$

EXEMPLUM 5

395. *Casus in hac forma* $\int \frac{x^{p-1}dx}{\sqrt[6]{(1-x^6)^{6-q}}} = \left(\frac{p}{q}\right)$ *contentos, ubi* $n=6$, *evolvere.*

A circulo pendent hae tres formulae

$$\left(\frac{5}{1}\right) = \frac{\pi}{6 \sin. \frac{\pi}{6}} = \frac{\pi}{3} = \alpha, \quad \left(\frac{4}{2}\right) = \frac{\pi}{6 \sin. \frac{2\pi}{6}} = \frac{\pi}{3\sqrt{3}} = \beta, \quad \left(\frac{3}{3}\right) = \frac{\pi}{6 \sin. \frac{3\pi}{6}} = \frac{\pi}{6} = \gamma;$$

tum vero assumantur hae duae transcendentes

$$\left(\frac{4}{1}\right) = A \quad \text{et} \quad \left(\frac{3}{2}\right) = B$$

atque per has omnes sequenti modo determinantur

$$\left(\frac{6}{1}\right) = 1, \quad \left(\frac{6}{2}\right) = \frac{1}{2}, \quad \left(\frac{6}{3}\right) = \frac{1}{3}, \quad \left(\frac{6}{4}\right) = \frac{1}{4}, \quad \left(\frac{6}{5}\right) = \frac{1}{5}, \quad \left(\frac{6}{6}\right) = \frac{1}{6},$$

$$\left(\frac{5}{1}\right) = \alpha, \quad \left(\frac{5}{2}\right) = \frac{\beta}{A}, \quad \left(\frac{5}{3}\right) = \frac{\gamma}{2B}, \quad \left(\frac{5}{4}\right) = \frac{\beta}{3B}, \quad \left(\frac{5}{5}\right) = \frac{\alpha}{4A},$$

$$\left(\frac{4}{1}\right) = A, \quad \left(\frac{4}{2}\right) = \beta, \quad \left(\frac{4}{3}\right) = \frac{\beta\gamma}{\alpha B}, \quad \left(\frac{4}{4}\right) = \frac{\beta\gamma A}{2\alpha BB},$$

$$\left(\frac{3}{1}\right) = \frac{\alpha B}{\beta}, \quad \left(\frac{3}{2}\right) = B, \quad \left(\frac{3}{3}\right) = \gamma,$$

$$\left(\frac{2}{1}\right) = \frac{\alpha B}{\gamma}, \quad \left(\frac{2}{2}\right) = \frac{\alpha BB}{\gamma A},$$

$$\left(\frac{1}{1}\right) = \frac{\alpha A}{\beta}.$$

SCHOLION

396. Has determinationes, quousque libuerit, continuare licet, in quibus praecipue notari debent casus novas transcendentium species introducentes; quorum primus occurrit, si $n=3$, estque

$$\left(\frac{1}{1}\right) = \int \frac{dx}{\sqrt[3]{(1-x^3)^2}},$$

cuius valorem per productum infinitum supra [§ 371] vidimus esse

$$= \frac{3}{1} \cdot \frac{2 \cdot 6}{4 \cdot 4} \cdot \frac{5 \cdot 9}{7 \cdot 7} \cdot \frac{8 \cdot 12}{10 \cdot 10} \cdot \text{etc.},$$

quod ex formula $\left(\frac{1}{1}\right)$ ob $n=3$ etiam est

$$\frac{2}{1\cdot 1}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{6\cdot 8}{7\cdot 7}\cdot\frac{9\cdot 11}{10\cdot 10}\cdot\frac{12\cdot 14}{13\cdot 13}\cdot\text{etc.}$$

Deinde ex classe $n=4$ nascitur haec nova forma transcendens

$$\left(\frac{2}{1}\right)=\int\frac{x\,dx}{\sqrt[4]{(1-x^4)^3}}=\int\frac{dx}{\sqrt[4]{(1-x^4)^2}}=\int\frac{dx}{\sqrt{(1-x^4)}},$$

quae aequatur huic producto infinito

$$\frac{3}{1\cdot 2}\cdot\frac{4\cdot 7}{5\cdot 6}\cdot\frac{8\cdot 11}{9\cdot 10}\cdot\frac{12\cdot 15}{13\cdot 14}\cdot\frac{16\cdot 19}{17\cdot 18}\cdot\text{etc.}\quad=\frac{3}{2}\cdot\frac{2\cdot 7}{5\cdot 3}\cdot\frac{4\cdot 11}{9\cdot 5}\cdot\frac{6\cdot 15}{13\cdot 7}\cdot\frac{8\cdot 19}{17\cdot 9}\cdot\text{etc.}$$

Ex classe $n=5$ impetramus duas novas formulas transcendentes

$$\left(\frac{3}{1}\right)=\int\frac{x^2dx}{\sqrt[5]{(1-x^5)^4}}=\int\frac{dx}{\sqrt[5]{(1-x^5)^2}}=\frac{4}{1\cdot 3}\cdot\frac{5\cdot 9}{6\cdot 8}\cdot\frac{10\cdot 14}{11\cdot 13}\cdot\frac{15\cdot 19}{16\cdot 18}\cdot\text{etc.}$$

et

$$\left(\frac{2}{2}\right)=\int\frac{x\,dx}{\sqrt[5]{(1-x^5)^3}}=\frac{4}{2\cdot 2}\cdot\frac{5\cdot 9}{7\cdot 7}\cdot\frac{10\cdot 14}{12\cdot 12}\cdot\frac{15\cdot 19}{17\cdot 17}\cdot\text{etc.},$$

ita ut sit

$$\left(\frac{3}{1}\right):\left(\frac{2}{2}\right)=\frac{2\cdot 2}{1\cdot 3}\cdot\frac{7\cdot 7}{6\cdot 8}\cdot\frac{12\cdot 12}{11\cdot 13}\cdot\frac{17\cdot 17}{16\cdot 18}\cdot\text{etc.}$$

Classis $n=6$ has duas formulas transcendentes suppeditat

$$\left(\frac{4}{1}\right)=\int\frac{x^3dx}{\sqrt[6]{(1-x^6)^5}}=\int\frac{dx}{\sqrt[3]{(1-x^6)}}=\frac{1}{2}\int\frac{y\,dy}{\sqrt[6]{(1-y^3)^5}}$$

posito $xx=y$ et

$$\left(\frac{3}{2}\right)=\int\frac{x^2dx}{\sqrt[3]{(1-x^6)^2}}=\int\frac{x\,dx}{\sqrt{(1-x^6)}}=\frac{1}{2}\int\frac{dy}{\sqrt{(1-y^3)}}=\frac{1}{3}\int\frac{dz}{\sqrt[3]{(1-zz)^2}}$$

sumto $y=xx$ et $z=x^3$. Notandum autem est inter has et primam

$$\int\frac{dx}{\sqrt[3]{(1-x^3)^2}}=2\int\frac{y\,dy}{\sqrt[6]{(1-y^6)^4}}=2\left(\frac{2}{2}\right)$$

relationem dari, quae est [§ 384]

$$\gamma\left(\frac{4}{1}\right)\left(\frac{2}{2}\right)=\alpha\left(\frac{3}{2}\right)\left(\frac{3}{2}\right),$$

ita ut prima admissa hic altera sufficiat.

CALCVLI INTEGRALIS
LIBER PRIOR.

PARS PRIMA
SEV
METHODVS INVESTIGANDI FVNCTIONES
VNIVS VARIABILIS EX DATA RELATIONE QVACVN-
QVE DIFFERENTIALIVM PRIMI GRADVS.

SECTIO SECVNDA
DE
INTEGRATIONE AEQVATIONVM
DIFFERENTIALIVM.

CAPUT I

DE SEPARATIONE VARIABILIUM

DEFINITIO

397. *In aequatione differentiali separatio variabilium locum habere dicitur, cum aequationem ita in duo membra dispescere licet, ut in utroque unica tantum variabilis cum suo differentiali insit.*

COROLLARIUM 1

398. Quando igitur aequatio differentialis ita est comparata, ut ad hanc formam $Xdx = Ydy$ reduci possit, in qua X functio sit solius x et Y solius y, tum ea aequatio separationem variabilium admittere dicitur.

COROLLARIUM 2

399. Quodsi P et X functiones ipsius x tantum, at Q et Y functiones ipsius y tantum denotent, haec aequatio $PYdx = QXdy$ separationem variabilium admittit; nam per XY divisa abit in $\frac{Pdx}{X} = \frac{Qdy}{Y}$, in qua variabiles sunt separatae.

COROLLARIUM 3

400. In forma ergo generali $\frac{dy}{dx} = V$ separatio variabilium locum habet, si V eiusmodi fuerit functio ipsarum x et y, ut in duos factores resolvi possit, quorum alter solam variabilem x, alter solam y contineat. Si enim sit $V = XY$, inde prodit aequatio separata $\frac{dy}{Y} = Xdx$.

SCHOLION

401. Posita differentialium ratione $\frac{dy}{dx} = p$ in hac sectione eiusmodi relationem inter x, y et p considerare instituimus, qua p aequetur functioni cuicunque ipsarum x et y. Hic igitur primum eum casum contemplamur, quo ista functio in duos factores resolvitur, quorum alter est functio tantum ipsius x et alter ipsius y, ita ut aequatio ad hanc formam reduci possit $Xdx = Ydy$, in qua binae variabiles a se invicem separatae esse dicuntur. Atque in hoc casu formulae simplices ante tractatae continentur, quando $Y = 1$, ut sit $dy = Xdx$ et $y = \int Xdx$, ubi totum negotium ad integrationem formulae Xdx revocatur. Haud maiorem autem habet difficultatem aequatio separata $Xdx = Ydy$, quam perinde ac formulas simplices tractare licet, id quod in sequente problemate ostendemus.

PROBLEMA 49

402. *Aequationem differentialem, in qua variabiles sunt separatae, integrare seu aequationem inter ipsas variabiles invenire.*

SOLUTIO

Aequatio separationem variabilium admittens semper ad hanc formam $Ydy = Xdx$ reducitur, ubi Xdx tanquam differentiale functionis cuiusdam ipsius x et Ydy tanquam differentiale functionis cuiusdam ipsius y spectari potest. Cum igitur differentialia sint aequalia, eorum integralia quoque aequalia esse vel quantitate constante differre necesse est. Integrentur ergo per praecepta superioris sectionis seorsim ambae formulae seu quaerantur integralia $\int Ydy$ et $\int Xdx$, quibus inventis erit utique $\int Ydy = \int Xdx + \text{Const.}$, qua aequatione relatio finita inter quantitates x et y exprimitur.

COROLLARIUM 1

403. Quoties ergo aequatio differentialis separationem variabilium admittit, toties integratio per eadem praecepta, quae supra de formulis simplicibus sunt tradita, absolvi potest.

COROLLARIUM 2

404. In aequatione integrali $\int Ydy = \int Xdx +$ Const. vel ambae functiones $\int Ydy$ et $\int Xdx$ sunt algebraicae, vel altera algebraica, altera vero transcendens, vel ambae transcendentes, sicque relatio inter x et y vel erit algebraica vel transcendens.

SCHOLION

405. In separatione variabilium a nonnullis totum fundamentum resolutionis aequationum differentialium constitui solet, ita ut, cum aequatio proposita separationem variabilium non admittit, idonea substitutio sit investiganda, cuius beneficio novae variabiles introductae separationem patiantur. Totum ergo negotium huc reducitur, ut proposita aequatione differentiali quacunque eiusmodi substitutio seu novarum variabilium introductio doceatur, ut deinceps separatio variabilium locum sit habitura. Optandum utique esset, ut huiusmodi methodus pro quovis casu idoneam substitutionem inveniendi aperiretur; sed nihil omnino certi in hoc negotio est compertum, dum pleraeque substitutiones, quae adhuc in usu fuerunt, nullis certis principiis innituntur. Deinde autem variabilium separatio non tanquam verum fundamentum omnis integrationis spectari potest, propterea quod in aequationibus differentialibus secundi altiorisve gradus nullum usum praestat; infra autem aliud principium latissime patens sum expositurus. In hoc capite interim praecipuas integrationes ope separationis variabilium administratas exponere operae pretium videtur, quandoquidem in hoc arduo negotio quam plurimas methodos cognoscere plurimum interest.

PROBLEMA 50

406. *Aequationem differentialem $Pdx = Qdy$, in qua P et Q sint functiones homogeneae eiusdem dimensionum numeri ipsarum x et y, ad separationem variabilium reducere eiusque integrale invenire.*

SOLUTIO

Cum P et Q sint functiones homogeneae ipsarum x et y eiusdem dimensionum numeri, erit $\frac{P}{Q}$ functio homogenea nullius dimensionis, quae ergo posito $y = ux$ abit in functionem ipsius u. Ponatur igitur $y = ux$ abeatque $\frac{P}{Q}$ in U functionem ipsius u, ita ut sit $dy = Udx$. Sed ob $y = ux$ fit $dy = udx + xdu$, qua substitutione nostra aequatio induet hanc formam $udx + xdu = Udx$ inter binas variabiles x et u, quae manifesto sunt separabiles. Nam dispositis terminis dx continentibus ad unam partem habetur $xdu = (U - u)dx$ ideoque

$$\frac{dx}{x} = \frac{du}{U-u},$$

quae integrata dat $lx = \int \frac{du}{U-u}$, ita ut iam ex variabili u determinetur x, unde porro cognoscitur $y = ux$.

COROLLARIUM 1

407. Quodsi ergo integrale $\int \frac{du}{U-u}$ etiam per logarithmos exprimi possit, ita ut lx aequetur logarithmo functionis cuiuspiam [algebraicae] ipsius u, habebitur aequatio algebraica inter x et u ideoque pro u posito valore $\frac{y}{x}$ aequatio algebraica inter x et y.

COROLLARIUM 2

408. Cum sit $y = ux$, erit $ly = lu + lx$ ideoque, cum sit $lx = \int \frac{du}{U-u}$, erit

$$ly = lu + \int \frac{du}{U-u} = \int \frac{du}{u} + \int \frac{du}{U-u},$$

quibus integralibus in unum reductis fit $ly = \int \frac{Udu}{u(U-u)}$. Verum hic notandum est non in utraque integratione pro lx et ly constantem arbitrariam adiicere licere; statim enim atque alteri integrali est adiecta, simul constans alteri adiicienda definitur, cum esse debeat $ly = lx + lu$.

COROLLARIUM 3

409. Cum sit

$$\int \frac{du}{U-u} = \int \frac{du - dU + dU}{U-u} = \int \frac{dU}{U-u} - \int \frac{dU - du}{U-u},$$

ob hoc posterius membrum per logarithmos integrabile erit

$$lx = \int \frac{dU}{U-u} - l(U-u) \quad \text{seu} \quad lx(U-u) = \int \frac{dU}{U-u}.$$

Perinde ergo est, sive haec formula $\int \frac{du}{U-u}$ sive $\int \frac{dU}{U-u}$ integretur.

SCHOLION

410. Quoniam haec methodus ad omnes aequationes homogeneas patet neque etiam ob irrationalitatem, quae forte in functionibus P et Q inest, impeditur, imprimis est aestimanda plurimumque aliis methodis anteferenda, quae tantum ad aequationes nimis speciales sunt accommodatae. Atque hinc etiam discimus omnes aequationes, quae ope cuiusdam substitutionis ad homogeneitatem revocari possunt, per eandem methodum tractari posse. Veluti si proponatur haec aequatio

$$dz + zzdx = \frac{adx}{xx},$$

statim patet posito $z = \frac{1}{y}$ eam ad hanc homogeneam $-\frac{dy}{yy} + \frac{dx}{yy} = \frac{adx}{xx}$ seu

$$xxdy = dx(xx - ayy)$$

reduci [§ 414].

Caeterum non difficulter perspicitur, utrum aequatio proposita huiusmodi substitutione ad homogeneitatem perduci queat. Plerumque, quoties quidem fieri potest, sufficit has positiones $x = u^m$ et $y = v^n$ tentasse, ubi facile iudicabitur, num exponentes m et n ita assumere liceat, ut ubique idem dimensionum numerus prodeat; magis enim complicatis substitutionibus in hoc genere vix locus conceditur, nisi forte quasi sponte se prodant. Methodum autem integrandi hic expositam aliquot exemplis illustrasse iuvabit.

EXEMPLUM 1

411. *Proposita aequatione differentiali homogenea* $xdx + ydy = mydx$ *eius integrale invenire.*

Cum ergo hinc sit $\frac{dy}{dx} = \frac{my - x}{y}$, posito $y = ux$ fit $\frac{my - x}{y} = \frac{mu - 1}{u}$ ideoque ob $dy = udx + xdu$ erit

$$udx + xdu = \frac{mu - 1}{u} dx$$

hincque

$$\frac{dx}{x}=\frac{udu}{mu-1-uu}=\frac{-udu}{1-mu+uu}$$

seu

$$\frac{dx}{x}=\frac{-udu+\frac{1}{2}mdu}{1-mu+uu}-\frac{\frac{1}{2}mdu}{1-mu+uu},$$

unde integrando

$$lx=-\frac{1}{2}l(1-mu+uu)-\frac{1}{2}m\int\frac{du}{1-mu+uu}+\text{Const.},$$

ubi tres casus sunt considerandi, prout $m>2$ vel $m<2$ vel $m=2$.

1) Sit $m>2$ et $1-mu+uu$ huiusmodi formam habebit

$$\left(u-a\right)\left(u-\frac{1}{a}\right),$$

ut sit $m=a+\frac{1}{a}=\frac{aa+1}{a}$, et ob

$$\frac{du}{\left(u-a\right)\left(u-\frac{1}{a}\right)}=\frac{a}{aa-1}\cdot\frac{du}{u-a}-\frac{a}{aa-1}\cdot\frac{du}{u-\frac{1}{a}}$$

fiet

$$lx=-\frac{1}{2}l(1-mu+uu)-\frac{aa+1}{2(aa-1)}\,l\frac{u-a}{u-\frac{1}{a}}+C$$

seu

$$lx\sqrt{(1-mu+uu)}+\frac{aa+1}{2(aa-1)}\,l\frac{au-aa}{au-1}=lc$$

et restituto valore $u=\frac{y}{x}$ aequatio integralis erit

$$l\sqrt{(xx-mxy+yy)}+\frac{aa+1}{2(aa-1)}\,l\frac{ay-aax}{ay-x}=lc$$

seu

$$\left(\frac{ay-aax}{ay-x}\right)^{\frac{aa+1}{2(aa-1)}}\sqrt{(xx-mxy+yy)}=c.$$

2) Sit $m<2$ seu $m=2\cos.\alpha$; erit

$$\int\frac{du}{1-2u\cos.\alpha+uu}=\frac{1}{\sin.\alpha}\text{ Ang. tang. }\frac{u\sin.\alpha}{1-u\cos.\alpha},$$

unde

$$lx\sqrt{(1-mu+uu)}=C-\frac{\cos.\alpha}{\sin.\alpha}\text{ Ang. tang. }\frac{u\sin.\alpha}{1-u\cos.\alpha}$$

seu

$$l\sqrt{(xx-mxy+yy)}=C-\frac{\cos.\alpha}{\sin.\alpha}\text{ Ang. tang. }\frac{y\sin.\alpha}{x-y\cos.\alpha}.$$

3) Sit $m=2$; erit

$$\int\frac{du}{(1-u)^2}=\frac{1}{1-u}$$

hincque

$$lx(1-u)=C-\frac{1}{1-u}\quad\text{seu}\quad l(x-y)=C-\frac{x}{x-y}.$$

EXEMPLUM 2

412. *Proposita aequatione differentiali homogenea* $dx(\alpha x+\beta y)=dy(\gamma x+\delta y)$ *eius integrale invenire.*

Posito $y=ux$ erit $udx+xdu=dx\cdot\frac{\alpha+\beta u}{\gamma+\delta u}$ ideoque

$$\frac{dx}{x}=\frac{du(\gamma+\delta u)}{\alpha+\beta u-\gamma u-\delta uu}=\frac{du(\delta u+\frac{1}{2}\gamma-\frac{1}{2}\beta)+du(\frac{1}{2}\gamma+\frac{1}{2}\beta)}{\alpha+(\beta-\gamma)u-\delta uu},$$

unde integrando

$$lx=C-l\sqrt{(\alpha+(\beta-\gamma)u-\delta uu)}+\frac{1}{2}(\beta+\gamma)\int\frac{du}{\alpha+(\beta-\gamma)u-\delta uu},$$

ubi iidem casus qui ante sunt considerandi, prout scilicet denominator $\alpha+(\beta-\gamma)u-\delta uu$ vel duos factores habet reales et inaequales vel aequales vel imaginarios.

EXEMPLUM 3

413. *Proposita aequatione differentiali homogenea* $xdx+ydy=xdy-ydx$ *eius integrale invenire.*

Cum hinc sit $\frac{dy}{dx}=\frac{x+y}{x-y}$, posito $y=ux$ fit $udx+xdu=\frac{1+u}{1-u}dx$ seu $xdu=\frac{1+uu}{1-u}dx$, unde colligitur

$$\frac{dx}{x}=\frac{du-udu}{1+uu}$$

et integrando

$$lx = \text{Ang. tang. } u - l\sqrt{(1+uu)} + C$$

seu

$$l\sqrt{(xx+yy)} = C + \text{Ang. tang. } \frac{y}{x}.$$

EXEMPLUM 4

414. *Proposita aequatione differentiali homogenea* $xxdy = (xx - ayy)dx$ *eius integrale invenire.*

Hic ergo est $\frac{dy}{dx} = \frac{xx - ayy}{xx}$ et posito $y = ux$ prodit

$$udx + xdu = (1 - auu)dx$$

ideoque

$$\frac{dx}{x} = \frac{du}{1 - u - auu} \quad \text{et} \quad lx = \int \frac{du}{1 - u - auu},$$

cuius evolutioni non opus est immorari.

EXEMPLUM 5

415. *Proposita aequatione differentiali homogenea* $xdy - ydx = dx\sqrt{(xx+yy)}$ *eius integrale invenire.*

Erit ergo $\frac{dy}{dx} = \frac{y + \sqrt{(xx+yy)}}{x}$, unde posito $y = ux$ fit

$$udx + xdu = \left(u + \sqrt{(1+uu)}\right)dx$$

seu

$$xdu = dx\sqrt{(1+uu)},$$

ita ut sit

$$\frac{dx}{x} = \frac{du}{\sqrt{(1+uu)}},$$

cuius integrale est

$$lx = la + l\left(u + \sqrt{(1+uu)}\right) = la + l\left(\frac{y + \sqrt{(xx+yy)}}{x}\right)$$

seu

$$lx = la + l\frac{x}{\sqrt{(xx+yy)} - y},$$

unde colligitur $x = \frac{ax}{\sqrt{(xx+yy)} - y}$ seu $\sqrt{(xx+yy)} = a + y$ hincque

$$xx = aa + 2ay.$$

SCHOLION

416. Huc etiam functiones transcendentes numerari possunt, modo afficiant functiones nullius dimensionis ipsarum x et y, quia posito $y = ux$ simul in functiones ipsius u abeunt. Ita si in aequatione $Pdx = Qdy$, praeterquam quod P et Q sunt functiones homogeneae eiusdem dimensionum numeri, insint huiusmodi formulae $l\frac{\sqrt{(xx+yy)}}{x}$, $e^{y:x}$, Ang. sin. $\frac{x}{\sqrt{(xx+yy)}}$, cos. $\frac{nx}{y}$ etc., methodus exposita pari successu adhiberi potest, quia posito $y = ux$ ratio $\frac{dy}{dx}$ aequatur functioni solius novae variabilis u.

PROBLEMA 51

417. *Aequationem differentialem primi ordinis*

$$dx(\alpha + \beta x + \gamma y) = dy(\delta + \varepsilon x + \zeta y)$$

ad separationem variabilium revocare et integrare.

SOLUTIO

Ponatur

$$\alpha + \beta x + \gamma y = t \quad \text{et} \quad \delta + \varepsilon x + \zeta y = u,$$

ut fiat $tdx = udy$. At inde colligimus

$$x = \frac{\zeta t - \gamma u - \alpha\zeta + \gamma\delta}{\beta\zeta - \gamma\varepsilon} \quad \text{et} \quad y = \frac{\beta u - \varepsilon t + \alpha\varepsilon - \beta\delta}{\beta\zeta - \gamma\varepsilon}$$

hincque

$$dx : dy = \zeta dt - \gamma du : \beta du - \varepsilon dt,$$

unde nanciscimur hanc aequationem

$$\zeta tdt - \gamma tdu = \beta udu - \varepsilon udt$$

seu

$$dt(\zeta t + \varepsilon u) = du(\beta u + \gamma t);$$

quae cum sit homogenea et cum exemplo § 412 conveniat, integratio iam est expedita.

Verum tamen casus existit, quo haec reductio ad homogeneitatem locum non habet, cum fuerit $\beta\zeta - \gamma\varepsilon = 0$, quoniam tum introductio novarum variabilium t et u tollitur. Hic ergo casus peculiarem requirit solutionem, quae ita instituatur. Quoniam tum aequatio proposita eiusmodi formam est habitura

$$\alpha dx + (\beta x + \gamma y)dx = \delta dy + n(\beta x + \gamma y)dy,$$

ponamus $\beta x + \gamma y = z$; erit

$$\frac{dy}{dx} = \frac{\alpha + z}{\delta + nz}.$$

At $dy = \frac{dz - \beta dx}{\gamma}$, ergo

$$\frac{dz - \beta dx}{\gamma} = \frac{\alpha + z}{\delta + nz} dx,$$

ubi variabiles manifesto sunt separabiles; fit enim

$$dx = \frac{dz(\delta + nz)}{\alpha\gamma + \beta\delta + (\gamma + n\beta)z},$$

cuius integratio logarithmos involvit, nisi sit $\gamma + n\beta = 0$, quo casu algebraice dat $x = \frac{2\delta z + nzz}{2(\alpha\gamma + \beta\delta)} + C$.

COROLLARIUM 1

418. Aequatio ergo differentialis primi ordinis, uti vocatur, in genere ad homogeneitatem reduci nequit, sed casus, quibus $\beta\zeta = \gamma\varepsilon$, inde excipi debent, qui etiam ad aequationem separatam omnino diversam deducunt.

COROLLARIUM 2

419. Si in his casibus exceptis sit $n = 0$ seu haec proposita sit aequatio $dy = dx(\alpha + \beta x + \gamma y)$, posito $\beta x + \gamma y = z$ ob $\delta = 1$ haec oritur aequatio $dx = \frac{dz}{\alpha\gamma + \beta + \gamma z}$, cuius integrale est

$$\gamma x = l\frac{\beta + \alpha\gamma + \gamma z}{C} = l\frac{\beta + \alpha\gamma + \beta\gamma x + \gamma\gamma y}{C}$$

seu

$$\beta + \gamma(\alpha + \beta x + \gamma y) = Ce^{\gamma x}.$$

PROBLEMA 52

420. *Proposita aequatione differentiali huiusmodi*

$$dy + Pydx = Qdx,$$

in qua P et Q sint functiones quaecunque ipsius x, altera autem variabilis y cum suo differentiali nusquam plus una habeat dimensionem, eam ad separationem variabilium perducere et integrare.

SOLUTIO

Quaeratur eiusmodi functio ipsius x, quae sit X, ut facta substitutione $y = Xu$ aequatio prodeat separabilis. Tum autem oritur

$$Xdu + udX + PXudx = Qdx,$$

quam aequationem separationem admittere evidens est, si fuerit $dX + PXdx = 0$ seu

$$\frac{dX}{X} = -Pdx,$$

unde integratio dat

$$lX = -\int Pdx \quad \text{et} \quad X = e^{-\int Pdx};$$

hac ergo pro X sumta functione aequatio nostra transformata erit $Xdu = Qdx$ seu

$$du = \frac{Qdx}{X} = e^{\int Pdx} Qdx,$$

unde, cum P et Q sint functiones datae ipsius x, erit

$$u = \int e^{\int Pdx} Qdx = \frac{y}{X}.$$

Quocirca aequationis propositae integrale est

$$y = e^{-\int Pdx} \int e^{\int Pdx} Qdx.$$

COROLLARIUM 1

421. Resolutio ergo huius aequationis $dy + Pydx = Qdx$ duplicem requirit integrationem, alteram formulae $\int Pdx$, alteram formulae $\int e^{\int Pdx} Qdx$.

Sufficit autem in posteriori constantem arbitrariam adiecisse, cum valor ipsius y plus una non recipiat. Etiamsi enim in priori loco $\int Pdx$ scribatur $\int Pdx + C$, formula pro y manet eadem.

COROLLARIUM 2

422. Dum ergo formula Pdx integratur, sufficit eius integrale particulare sumi ideoque constanti ingredienti eiusmodi valorem tribui convenit, ut integralis forma fiat simplicissima.

SCHOLION

423. En ergo aliud aequationum genus non minus late patens quam praecedens homogenearum, quod ad separationem variabilium perduci hocque modo integrari potest. Inde autem in Analysin maxima utilitas redundat, cum hic litterae P et Q functiones quascunque ipsius x denotent. Hoc ergo modo manifestum est tractari posse hanc aequationem

$$Rdy + Pydx = Qdx,$$

si etiam R functionem quamcunque ipsius x denotet. Facta enim divisione per R forma proposita prodit, modo loco P et Q scribatur $\frac{P}{R}$ et $\frac{Q}{R}$, ita ut integrale futurum sit

$$y = e^{-\int \frac{Pdx}{R}} \int \frac{e^{\int \frac{Pdx}{R}} Qdx}{R}.$$

Ad huius problematis illustrationem quaedam exempla adiiciamus.

EXEMPLUM 1

424. *Proposita aequatione differentiali $dy + ydx = x^n dx$ eius integrale invenire.*

Cum hic sit $P = 1$ et $Q = x^n$, erit $\int Pdx = x$ et aequatio integralis fiet

$$y = e^{-x} \int e^x x^n dx,$$

quae, si n sit numerus integer positivus, evadet [§ 223]

$$y = e^{-x}(e^x(x^n - nx^{n-1} + n(n-1)x^{n-2} - \text{etc.}) + C),$$

qua evoluta prodit

$$y = Ce^{-x} + x^n - nx^{n-1} + n(n-1)x^{n-2} - n(n-1)(n-2)x^{n-3} + \text{etc.},$$

unde pro simplicioribus valoribus ipsius n,

$$\begin{aligned} &\text{si } n = 0, \quad \text{erit} \quad y = Ce^{-x} + 1, \\ &\text{si } n = 1, \quad \text{erit} \quad y = Ce^{-x} + x - 1, \\ &\text{si } n = 2, \quad \text{erit} \quad y = Ce^{-x} + x^2 - 2x + 2 \cdot 1, \\ &\text{si } n = 3, \quad \text{erit} \quad y = Ce^{-x} + x^3 - 3x^2 + 3 \cdot 2x - 3 \cdot 2 \cdot 1 \\ &\qquad\qquad \text{etc.} \end{aligned}$$

COROLLARIUM 1

425. Si ergo constans C sumatur $= 0$, habebitur integrale particulare

$$y = x^n - nx^{n-1} + n(n-1)x^{n-2} - n(n-1)(n-2)x^{n-3} + \text{etc.},$$

quod ergo est algebraicum, dummodo n sit numerus integer positivus.

COROLLARIUM 2

426. Si integrale ita determinari debeat, ut posito $x = 0$ valor ipsius y evanescat, constans C aequalis sumi debet ultimo termino constanti signo mutato, unde id semper erit transcendens.

EXEMPLUM 2

427. *Proposita aequatione differentiali* $(1 - xx)dy + xydx = adx$ *eius integrale invenire.*

Aequatio ista per $1 - xx$ divisa ad hanc formam reducitur

$$dy + \frac{xydx}{1-xx} = \frac{adx}{1-xx},$$

ita ut sit $P = \frac{x}{1-xx}$, $Q = \frac{a}{1-xx}$, hinc

$$\int Pdx = -l\sqrt{(1-xx)} \quad \text{et} \quad e^{\int Pdx} = \frac{1}{\sqrt{(1-xx)}},$$

ex quo integrale reperitur

$$y = \sqrt{(1-xx)}\int\frac{a\,dx}{(1-xx)^{\frac{3}{2}}} = \left(\frac{ax}{\sqrt{(1-xx)}} + C\right)\sqrt{(1-xx)},$$

quocirca integrale quaesitum erit

$$y = ax + C\sqrt{(1-xx)};$$

quod si ita determinari debeat, ut posito $x = 0$ [evanescat], sumi oportet $C = 0$ eritque $y = ax$.

EXEMPLUM 3

428. *Proposita aequatione differentiali* $dy + \frac{ny\,dx}{\sqrt{(1+xx)}} = a\,dx$ *eius integrale invenire.*

Cum hic sit $P = \frac{n}{\sqrt{(1+xx)}}$ et $Q = a$, erit

$$\int P\,dx = nl(x + \sqrt{(1+xx)}) \quad \text{et} \quad e^{\int P\,dx} = (x + \sqrt{(1+xx)})^n$$

et

$$e^{-\int P\,dx} = \left(\sqrt{(1+xx)} - x\right)^n,$$

unde integrale quaesitum erit

$$y = \left(\sqrt{(1+xx)} - x\right)^n \int a\,dx\left(x + \sqrt{(1+xx)}\right)^n,$$

ad quod evolvendum ponatur $x + \sqrt{(1+xx)} = u$ et fiet $x = \frac{uu-1}{2u}$, hinc

$$dx = \frac{du(1+uu)}{2uu},$$

ergo

$$\int u^n dx = \frac{u^{n-1}}{2(n-1)} + \frac{u^{n+1}}{2(n+1)} + C.$$

Nunc quia $\left(\sqrt{(1+xx)} - x\right)^n = u^{-n}$, erit

$$y = Cu^{-n} + \frac{au^{-1}}{2(n-1)} + \frac{au}{2(n+1)}$$

sive

$$y = C\left(\sqrt{(1+xx)} - x\right)^n + \frac{a}{2(n-1)}\left(\sqrt{(1+xx)} - x\right) + \frac{a}{2(n+1)}\left(\sqrt{(1+xx)} + x\right),$$

quae expressio ad hanc formam reducitur

$$y = C\left(\sqrt{(1+xx)} - x\right)^n + \frac{na}{nn-1}\sqrt{(1+xx)} - \frac{ax}{nn-1}.$$

Si integrale ita determinari debeat, ut posito $x = 0$ fiat $y = 0$, sumi oportet $C = -\frac{na}{nn-1}$.

PROBLEMA 53

429. *Proposita aequatione differentiali*

$$dy + Pydx = Qy^{n+1}dx,$$

ubi P et Q denotent functiones quascunque ipsius x, eam ad separationem variabilium reducere et integrare.

SOLUTIO

Haec aequatio posito $\frac{1}{y^n} = z$ statim ad formam modo tractatam reducitur; nam ob $\frac{dy}{y} = -\frac{dz}{nz}$ aequatio nostra per y divisa, scilicet

$$\frac{dy}{y} + Pdx = Qy^n dx,$$

statim abit in

$$-\frac{dz}{nz} + Pdx = \frac{Qdx}{z} \quad \text{seu} \quad dz - nPzdx = -nQdx,$$

cuius integrale est

$$z = -e^{n\int Pdx}\int e^{-n\int Pdx} nQdx$$

ideoque

$$\frac{1}{y^n} = -ne^{n\int Pdx}\int e^{-n\int Pdx} Qdx.$$

Tractari autem potest ut praecedens quaerendo eiusmodi functionem X, ut facta substitutione $y = Xu$ prodeat aequatio separabilis; prodit autem

$$Xdu + udX + PXudx = X^{n+1}u^{n+1}Qdx.$$

Fiat ergo

$$dX + PXdx = 0 \quad \text{seu} \quad X = e^{-\int Pdx}$$

eritque

$$\frac{du}{u^{n+1}} = X^n Qdx = e^{-n\int Pdx} Qdx$$

et integrando

$$-\frac{1}{nu^n}=\int e^{-n\int Pdx}Qdx.$$

Iam quia

$$u=\frac{y}{X}=e^{\int Pdx}y,$$

habebitur ut ante

$$\frac{1}{y^n}=-ne^{n\int Pdx}\int e^{-n\int Pdx}Qdx.$$

SCHOLION

430. Hic ergo casus a praecedente non differre est censendus, ita ut hic nihil novi sit praestitum. Atque haec duo genera sunt fere sola, quae quidem aliquanto latius pateant, in quibus separatio variabilium obtineri queat. Caeteri casus, qui ope cuiusdam substitutionis ad variabilium separationem praeparari possunt, plerumque sunt nimis speciales, quam ut insignis usus inde expectari possit. Interim tamen aliquot casus prae caeteris memorabiles hic exponamus.

PROBLEMA 54

431. *Proposita hac aequatione differentiali*

$$\alpha ydx+\beta xdy+x^m y^n(\gamma ydx+\delta xdy)=0$$

eam ad separationem variabilium reducere et integrare.

SOLUTIO

Tota aequatione per xy divisa nanciscimur hanc formam

$$\frac{\alpha dx}{x}+\frac{\beta dy}{y}+x^m y^n\left(\frac{\gamma dx}{x}+\frac{\delta dy}{y}\right)=0,$$

unde statim has substitutiones $x^\alpha y^\beta=t$ et $x^\gamma y^\delta=u$ insigni usu non esse carituras colligimus; inde enim fit

$$\frac{\alpha dx}{x}+\frac{\beta dy}{y}=\frac{dt}{t} \quad \text{et} \quad \frac{\gamma dx}{x}+\frac{\delta dy}{y}=\frac{du}{u}$$

hincque aequatio nostra

$$\frac{dt}{t} + x^m y^n \frac{du}{u} = 0.$$

At ex substitutione sequitur $x^{\alpha\delta-\beta\gamma} = t^\delta u^{-\beta}$ et $y^{\alpha\delta-\beta\gamma} = u^\alpha t^{-\gamma}$ ideoque

$$x = t^{\frac{\delta}{\alpha\delta-\beta\gamma}} u^{\frac{-\beta}{\alpha\delta-\beta\gamma}} \quad \text{et} \quad y = t^{\frac{-\gamma}{\alpha\delta-\beta\gamma}} u^{\frac{\alpha}{\alpha\delta-\beta\gamma}},$$

quibus substitutis fit

$$\frac{dt}{t} + t^{\frac{\delta m-\gamma n}{\alpha\delta-\beta\gamma}} u^{\frac{\alpha n-\beta m}{\alpha\delta-\beta\gamma}} \frac{du}{u} = 0$$

ideoque

$$t^{\frac{\gamma n-\delta m}{\alpha\delta-\beta\gamma}-1} dt + u^{\frac{\alpha n-\beta m}{\alpha\delta-\beta\gamma}-1} du = 0,$$

cuius aequationis integrale est

$$\frac{t^{\frac{\gamma n-\delta m}{\alpha\delta-\beta\gamma}}}{\gamma n-\delta m} + \frac{u^{\frac{\alpha n-\beta m}{\alpha\delta-\beta\gamma}}}{\alpha n-\beta m} = C,$$

ubi tantum superest, ut restituantur valores $t = x^\alpha y^\beta$ et $u = x^\gamma y^\delta$. Caeterum notetur, si fuerit vel $\gamma n - \delta m = 0$ vel $\alpha n - \beta m = 0$, loco illorum membrorum vel lt vel lu scribi debere.

SCHOLION

432. Ad aequationem propositam ducit quaestio, qua eiusmodi relatio inter variabiles x et y quaeritur, ut fiat

$$\int y\,dx = axy + bx^{m+1}y^{n+1};$$

ad hanc enim resolvendam differentialia sumi debent, quo prodit

$$y\,dx = ax\,dy + ay\,dx + bx^m y^n((m+1)y\,dx + (n+1)x\,dy),$$

qua aequatione cum nostra forma comparata est

$$\alpha = a-1, \quad \beta = a, \quad \gamma = (m+1)b \quad \text{et} \quad \delta = (n+1)b,$$

ergo

$$\alpha\delta - \beta\gamma = (n-m)ab - (n+1)b,$$

$$\alpha n - \beta m = (n-m)a - n \quad \text{et} \quad \gamma n - \delta m = (n-m)b,$$

unde aequatio integralis fit manifesta.

PROBLEMA 55

433. *Proposita hac aequatione differentiali*

$$ydy + dy(a + bx + nxx) = ydx(c + nx)$$

eam ad separationem variabilium reducere et integrare.

SOLUTIO

Cum hinc sit

$$\frac{dy}{dx} = \frac{y(c+nx)}{y+a+bx+nxx},$$

tentetur haec substitutio

$$\frac{y(c+nx)}{y+a+bx+nxx} = u \quad \text{seu} \quad y = \frac{u(a+bx+nxx)}{c+nx-u}$$

fierique debet $dy = udx$ seu

$$\frac{dy}{y} = \frac{udx}{y} = \frac{dx(c+nx-u)}{a+bx+nxx}.$$

At ex logarithmis colligitur

$$\frac{dy}{y} = \frac{du}{u} + \frac{dx(b+2nx)}{a+bx+nxx} - \frac{ndx-du}{c+nx-u} = \frac{dx(c+nx-u)}{a+bx+nxx},$$

quae contrahitur in

$$\frac{du(c+nx)-nudx}{u(c+nx-u)} = \frac{dx(c-b-nx-u)}{a+bx+nxx}$$

seu

$$\frac{du(c+nx)}{u(c+nx-u)} = \frac{dx(na+cc-bc+(b-2c)u+uu)}{(c+nx-u)(a+bx+nxx)},$$

quae per $c + nx - u$ multiplicata manifesto est separabilis, proditque

$$\frac{dx}{(a+bx+nxx)(c+nx)} = \frac{du}{u(na+cc-bc+(b-2c)u+uu)},$$

cuius ergo integratio per logarithmos et angulos absolvi potest. Casu autem hic vix praevidendo evenit, ut haec substitutio ad votum successerit, neque hoc problema magnopere iuvabit.

PROBLEMA 56

434. *Propositam hanc aequationem differentialem*

$$(y-x)dy = \frac{ndx(1+yy)\sqrt{(1+yy)}}{\sqrt{(1+xx)}}$$

ad separationem variabilium reducere et integrare.

SOLUTIO

Ob irrationalitatem duplicem vix ullo modo patet, cuiusmodi substitutione uti conveniat. Eiusmodi certe quaeri convenit, qua eidem signo radicali non ambae variabiles simul implicentur. Ad hunc scopum commoda videtur haec substitutio

$$y = \frac{x-u}{1+xu},$$

qua fit

$$y-x = \frac{-u(1+xx)}{1+xu}, \quad 1+yy = \frac{(1+xx)(1+uu)}{(1+xu)^2}$$

et

$$dy = \frac{dx(1+uu)-du(1+xx)}{(1+xu)^2},$$

atque his valoribus in nostra aequatione substitutis prodit

$$-udx(1+uu)+udu(1+xx) = ndx(1+uu)\sqrt{(1+uu)},$$

quae manifesto separationem variabilium admittit; colligitur scilicet

$$\frac{dx}{1+xx} = \frac{udu}{(1+uu)(n\sqrt{(1+uu)}+u)},$$

quae aequatio posito $1+uu=tt$ concinnior redditur

$$\frac{dx}{1+xx} = \frac{dt}{t(nt+\sqrt{(tt-1)})}$$

et ope positionis $t = \frac{1+ss}{2s}$ sublata irrationalitate

$$\frac{dx}{1+xx} = -\frac{2ds(1-ss)}{(1+ss)(n+1+(n-1)ss)} = -\frac{2ds}{1+ss}+\frac{2nds}{n+1+(n-1)ss},$$

cuius integratio nulla amplius laborat difficultate.

SCHOLION

435. In hoc casu praecipue substitutio $y = \frac{x - u}{1 + xu}$ notari meretur, qua duplex irrationalitas tollitur, unde operae pretium erit videre, quid hac substitutione generaliori praestari possit

$$y = \frac{\alpha x + u}{1 + \beta x u};$$

inde autem fit

$$\alpha - \beta yy = \frac{(\alpha - \beta uu)(1 - \alpha\beta xx)}{(1 + \beta xu)^2}, \quad y - \alpha x = \frac{u(1 - \alpha\beta xx)}{1 + \beta xu}$$

et

$$dy = \frac{dx(\alpha - \beta uu) + du(1 - \alpha\beta xx)}{(1 + \beta xu)^2}$$

ac iam facile perspicitur, in cuiusmodi aequationibus haec substitutio usum afferre possit; eius scilicet benificio haec duplex irrationalitas $\frac{\sqrt{(\alpha - \beta yy)}}{\sqrt{(1 - \alpha\beta xx)}}$ reducitur ad hanc simplicem $\frac{\sqrt{(\alpha - \beta uu)}}{1 + \beta xu}$, quam porro facile rationalem reddere licet.

Atque hi fere sunt casus, in quibus reductio ad separabilitatem locum invenit, quibus probe perpensis aditus facile patebit ad reliquos casus, qui quidem etiamnum sunt tractati; unicam vero adhuc investigationem apponam circa casus, quibus haec aequatio $dy + yydx = ax^m dx$ separationem variabilium admittit, quandoquidem ad huiusmodi aequationes frequenter pervenitur atque haec ipsa aequatio olim inter Geometras omni studio est agitata [§ 441].

PROBLEMA 57

436. *Pro aequatione* $dy + yydx = ax^m dx$ *valores exponentis* m *definire, quibus eam ad separationem variabilium reducere licet.*

SOLUTIO

Primo haec aequatio sponte est separabilis casu $m = 0$; tum enim ob $dy = dx(a - yy)$ fit $dx = \frac{dy}{a - yy}$. Omnis ergo investigatio in hoc versatur, ut ope substitutionum alii casus ad hunc reducantur.

Ponamus $y = \frac{b}{z}$ et fit

$$-bdz + bbdx = ax^m zzdx;$$

quae forma ut propositae similis evadat, statuatur $x^{m+1} = t$, ut sit

$$x^m dx = \frac{dt}{m+1} \quad \text{et} \quad dx = \frac{t^{\frac{-m}{m+1}} dt}{m+1},$$

eritque

$$bdz + \frac{azzdt}{m+1} = \frac{bb}{m+1} t^{\frac{-m}{m+1}} dt,$$

quae sumto $b = \frac{a}{m+1}$ ad similitudinem propositae propius accedit, ut sit

$$dz + zzdt = \frac{a}{(m+1)^2} t^{\frac{-m}{m+1}} dt.$$

Si ergo haec esset separabilis, ipsa proposita ista substitutione separabilis fieret et vicissim; unde concludimus, si aequatio proposita separationem admittat casu $m = n$, eam quoque esse admissuram casu $m = \frac{-n}{n+1}$. Hinc autem ex casu $m = 0$ alius non reperitur.

Ponamus $y = \frac{1}{x} - \frac{z}{xx}$, ut sit

$$dy = -\frac{dx}{xx} - \frac{dz}{xx} + \frac{2zdx}{x^3} \quad \text{et} \quad yydx = \frac{dx}{xx} - \frac{2zdx}{x^3} + \frac{zzdx}{x^4},$$

unde prodit

$$-\frac{dz}{xx} + \frac{zzdx}{x^4} = ax^m dx \quad \text{seu} \quad dz - \frac{zzdx}{xx} = -ax^{m+2}dx;$$

sit nunc $x = \frac{1}{t}$ et fit

$$dz + zzdt = at^{-m-4}dt;$$

quae cum propositae sit similis, discimus, si separatio succedat casu $m = n$, etiam succedere casu $m = -n - 4$.

Ex uno ergo casu $m = n$ consequimur duos, scilicet

$$m = -\frac{n}{n+1} \quad \text{et} \quad m = -n - 4.$$

Cum igitur constet casus $m = 0$, hinc formulae alternatim adhibitae praebent sequentes

$$m = -4,\quad m = -\frac{4}{3},\quad m = -\frac{8}{3},\quad m = -\frac{8}{5},\quad m = -\frac{12}{5},$$

$$m = -\frac{12}{7},\quad m = -\frac{16}{7}\quad \text{etc.},$$

qui casus omnes in hac formula $m = \frac{-4i}{2i \pm 1}$ continentur.

COROLLARIUM 1

437. Quodsi ergo fuerit vel

$$m = \frac{-4i}{2i+1} \quad \text{vel} \quad m = \frac{-4i}{2i-1},$$

aequatio $dy + yydx = ax^m dx$ per aliquot substitutiones repetitas[1]) tandem ad formam $du + uudv = cdv$, cuius separatio et integratio constat, reduci potest.

COROLLARIUM 2

438. Scilicet si fuerit $m = \frac{-4i}{2i+1}$, aequatio $dy + yydx = ax^m dx$ per substitutiones

$$x = t^{\frac{1}{m+1}} \quad \text{et} \quad y = \frac{a}{(m+1)z}$$

reducitur ad hanc

$$dz + zzdt = \frac{a}{(m+1)^2} t^n dt,$$

ut sit $n = \frac{-4i}{2i-1}$, qui casus uno gradu inferior est censendus.

1) Vide litteras ab EULERO ad GOLDBACHIUM 25. 11 (6. 12). 1731 et 3 (14). 1. 1732 scriptas (n. 788 indicis ENESTROEMIANI), Correspondance mathématique et physique publiée par P. H. FUSS, t. 1, St. Pétersbourg 1843, p. 58 et 63; *LEONHARDI EULERI Opera omnia*, series III, vol. 12. F. E.

COROLLARIUM 3

439. Sin autem fuerit $m = \frac{-4i}{2i-1}$, aequatio $dy + yydx = ax^m dx$ per has substitutiones

$$x = \frac{1}{t} \quad \text{et} \quad y = \frac{1}{x} - \frac{z}{xx} \quad \text{seu} \quad y = t - ttz$$

reducitur ad hanc $dz + zzdt = at^n dt$, in qua est

$$n = \frac{-4(i-1)}{2i-1} = \frac{-4(i-1)}{2(i-1)+1},$$

qui casus denuo uno gradu inferior est.

COROLLARIUM 4

440. Omnes ergo casus separabiles hoc modo inventi pro exponente m dant numeros negativos intra limites 0 et -4 contentos, ac si i sit numerus infinitus, prodit casus $m = -2$, qui autem per se constat, cum aequatio

$$dy + yydx = \frac{adx}{xx}$$

posito $y = \frac{1}{z}$ fiat homogenea [§ 410].

SCHOLION 1

441. Aequatio haec $dy + yydx = ax^m dx$ vocari solet RICCATIANA ab Auctore Comite RICCATI, qui primus casus separabiles proposuit.[1]) Hic quidem eam in forma simplicissima exhibui, cum eo haec $dy + Ayyt^\mu dt = Bt^\lambda dt$ ponendo $At^\mu dt = dx$ et $At^{\mu+1} = (\mu + 1)x$ statim reducatur.

Caeterum etsi binae substitutiones, quibus hic sum usus, sunt simplicissimae, tamen magis compositis adhibendis nulli alii casus separabiles deteguntur; ex quo hoc omnino memorabile est visum hanc aequationem rarissime separationem admittere, tametsi numerus casuum, quibus hoc praestari queat, revera sit infinitus.

1) IACOPO RICCATI (1676—1754) primus quidem proposuit problema casus separabiles inveniendi, Acta erud., Suppl. t. VIII, 1724, p. 73 et Acta erud. 1723, p. 509, sed DAN. BERNOULLI (1700—1782) primus hos casus publici iuris fecit, Acta erud. 1725, p. 473. F. E.

Caeterum haec investigatio ab exponente ad simplicem coefficientem traduci potest; posito enim $y = x^{\frac{m}{2}}z$ prodit

$$dz + \frac{mzdx}{2x} + x^{\frac{m}{2}}zzdx = ax^{\frac{m}{2}}dx,$$

ubi si fiat

$$x^{\frac{m}{2}}dx = dt \quad \text{et} \quad x^{\frac{m+2}{2}} = \frac{m+2}{2}t,$$

erit $\frac{dx}{x} = \frac{2dt}{(m+2)t}$ hincque

$$dz + \frac{mzdt}{(m+2)t} + zzdt = adt,$$

quae ergo aequatio, quoties fuerit $\frac{m}{m+2} = \pm 2i$ seu numerus par tam positivus quam negativus, separabilis reddi potest, ita ut haec aequatio

$$dz \pm \frac{2izdt}{t} + zzdt = adt$$

semper sit integrabilis. Si praeterea ponatur $z = u - \frac{m}{2(m+2)t}$, oritur

$$du + uudt = adt - \frac{m(m+4)dt}{4(m+2)^2tt}$$

et pro casibus separabilitatis $m = \frac{-4i}{2i \pm 1}$ habetur

$$du + uudt = adt + \frac{i(i \pm 1)dt}{tt}.$$

Uberiorem autem huius aequationis evolutionem, quandoquidem est maximi momenti, in sequentibus[1]) docebo, ubi de integratione aequationum differentialium per series infinitas sum acturus; hinc enim facilius casus separabiles eruemus simulque integralia assignare poterimus.

SCHOLION 2

442. Ampliora praecepta circa separationem variabilium, quae quidem usum sint habitura, vix tradi posse videntur, unde intelligitur in paucissimis aequationibus differentialibus hanc methodum adhiberi posse. Progrediar igitur ad aliud principium explicandum, unde integrationes haurire liceat,

1) Vide *Institutionum calculi integralis* vol. II cap. VII, § 940, 941, 943, 955—966; cf. quoque § 831—841; LEONHARDI EULERI *Opera omnia*, series I, vol. 12. F. E.

quod multo latius patet, dum etiam ad aequationes differentiales altiorum graduum accommodari potest, ita ut in eo verus ac naturalis fons omnium integrationum contineri videatur.

Istud autem principium in hoc consistit, quod proposita quacunque aequatione differentiali inter duas variabiles semper detur functio quaedam, per quam aequatio multiplicata fiat integrabilis; aequationis scilicet omnia membra ad eandem partem disponi oportet, ut talem formam obtineat $Pdx + Qdy = 0$; ac tum dico semper dari functionem quandam variabilium x et y, puta V, ut facta multiplicatione formula $VPdx + VQdy$ integrabilis existat seu ut verum sit differentiale ex differentiatione cuiuspiam functionis binarum variabilium x et y natum. Quodsi enim haec functio ponatur $= S$, ut sit $dS = VPdx + VQdy$, quia est $Pdx + Qdy = 0$, erit etiam $dS = 0$ ideoque $S = \text{Const.}$, quae ergo aequatio erit integrale idque completum aequationis differentialis $Pdx + Qdy = 0$. Totum ergo negotium ad inventionem illius multiplicatoris V redit.

CAPUT II

DE INTEGRATIONE AEQUATIONUM DIFFERENTIALIUM OPE MULTIPLICATORUM

PROBLEMA 58

443. *Propositam aequationem differentialem examinare, utrum per se sit integrabilis necne.*

SOLUTIO

Dispositis omnibus aequationis terminis ad eandem partem signi aequalitatis, ut huiusmodi habeatur forma $Pdx + Qdy = 0$, aequatio per se erit integrabilis, si formula $Pdx + Qdy$ fuerit verum differentiale functionis cuiuspiam binarum variabilium x et y. Hoc autem evenit, uti in *Calculo Differentiali* ostendimus,[1]) si differentiale ipsius P sumta sola y variabili ad dy eandem habeat rationem ac differentiale ipsius Q sumta sola x variabili ad dx, seu adhibito signandi modo, quo in *Calculo Differentiali* sumus usi, si fuerit

$$\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right).$$

Nam si Z sit ea functio, cuius differentiale est $Pdx + Qdy$, erit hoc signandi modo

$$P = \left(\frac{dZ}{dx}\right) \quad \text{et} \quad Q = \left(\frac{dZ}{dy}\right);$$

hinc ergo sequitur

$$\left(\frac{dP}{dy}\right) = \left(\frac{ddZ}{dx\,dy}\right) \quad \text{et} \quad \left(\frac{dQ}{dx}\right) = \left(\frac{ddZ}{dy\,dx}\right).$$

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1) Revera EULERUS hoc ibi non ostendit. Cf. *Institutiones calculi differentialis*, partis prioris § 231 et 240 (vide notam p. 41). F. E.

At est

$$\left(\frac{ddZ}{dxdy}\right) = \left(\frac{ddZ}{dydx}\right),$$

unde colligitur

$$\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right).$$

Quare proposita aequatione differentiali $Pdx + Qdy = 0$ utrum ea per se sit integrabilis necne, hoc modo dignoscetur. Quaerantur per differentiationem valores $\left(\frac{dP}{dy}\right)$ et $\left(\frac{dQ}{dx}\right)$, qui si fuerint inter se aequales, aequatio per se erit integrabilis; sin autem hi valores sint inaequales, aequatio non erit per se integrabilis.

COROLLARIUM 1

444. Omnes ergo aequationes differentiales, in quibus variabiles sunt a se invicem separatae, per se sunt integrabiles; habebunt enim huiusmodi formam $Xdx + Ydy = 0$, ut X sit functio solius x et Y solius y, eritque propterea $\left(\frac{dX}{dy}\right) = 0$ et $\left(\frac{dY}{dx}\right) = 0$.

COROLLARIUM 2

445. Vicissim igitur si proposita aequatione differentiali $Pdx + Qdy = 0$ fuerit $\left(\frac{dP}{dy}\right) = 0$ et $\left(\frac{dQ}{dx}\right) = 0$, variabiles in ea erunt separatae; littera enim P erit functio tantum ipsius x et Q tantum ipsius y. Unde aequationes separatae quasi primum genus aequationum per se integrabilium constituunt.

COROLLARIUM 3

446. Evidens autem est fieri posse, ut sit $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right)$, etiamsi neuter horum valorum sit nihilo aequalis. Dantur ergo aequationes per se integrabiles, licet variabiles in iis non sint separatae.

SCHOLION

447. Criterium hoc, quo aequationes per se integrabiles agnoscimus, maximi est momenti in hac, quam tradere suscipimus, methodo integrandi. Quodsi enim aequatio deprehendatur per se integrabilis, eius integrale per praecepta iam exposita inveniri potest; sin autem aequatio non fuerit per se

integrabilis, semper dabitur quantitas, per quam, si ea multiplicetur, fiat per se integrabilis; unde totum negotium eo revocabitur, ut proposita aequatione quacunque per se non integrabili inveniatur multiplicator idoneus, qui eam reddat per se integrabilem; qui si semper inveniri posset, nihil amplius in hac methodo integrandi esset desiderandum. Verum haec investigatio rarissime succedit ac vix adhuc latius patet quam ad eas aequationes, quas ope separationis variabilium iam tractare docuimus; interim tamen non dubito hanc methodum praecedenti longe praeferre, cum ad naturam aequationum magis videatur accommodata atque etiam ad aequationes differentiales altiorum graduum pateat, in quibus separatio variabilium nullius est usus.

PROBLEMA 59

448. *Aequationis differentialis, quam per se integrabilem esse constat, integrale invenire.*

SOLUTIO

Sit aequatio differentialis $Pdx + Qdy = 0$; in qua cum sit $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right)$, erit $Pdx + Qdy$ differentiale cuiuspiam functionis binarum variabilium x et y, quae sit Z, ut sit $dZ = Pdx + Qdy$. Cum ergo habeamus hanc aequationem $dZ = 0$, erit integrale quaesitum $Z = C$. Totum negotium ergo huc redit, ut ista functio Z eruatur, quod, cum sciamus esse $dZ = Pdx + Qdy$, haud difficulter praestabitur. Nam quia sumta tantum x variabili et altera y ut constante spectata est $dZ = Pdx$, habemus hic formulam differentialem simplicem unicam variabilem x involventem, quae per praecepta superioris sectionis integrata dabit $Z = \int Pdx +$ Const., ubi autem notandum est in hac constante quantitatem hic pro constanti habitam y utcunque inesse posse, unde eius loco scribatur Y, ut sit

$$Z = \int Pdx + Y.$$

Deinde simili modo x pro constante habeatur spectata sola y ut variabili, et cum sit $dZ = Qdy$, erit quoque $Z = \int Qdy +$ Const., quae constans autem quantitatem x involvet, ita ut sit functio ipsius x, qua posita X erit

$$Z = \int Qdy + X.$$

Quanquam autem neque hic functio X neque ibi functio Y determinatur, tamen, quia esse debet $\int Pdx + Y = \int Qdy + X$, hinc utraque determinabitur. Cum enim sit

$$\int Pdx - \int Qdy = X - Y,$$

haec quantitas $\int Pdx - \int Qdy$ semper in eiusmodi binas partes distinguetur, quarum altera est functio ipsius x tantum et altera ipsius y tantum, unde valores X et Y sponte cognoscuntur.

COROLLARIUM 1

449. Cum sit $Q = \left(\frac{dZ}{dy}\right)$, duplici integratione ne opus quidem est. Invento enim integrali $\int Pdx$ id iterum differentietur sumta sola y variabili prodeatque Vdy, unde necesse est fiat $Vdy + dY = Qdy$ ideoque

$$dY = Qdy - Vdy = (Q - V)dy.$$

COROLLARIUM 2

450. Aequationum ergo per se integrabilium $Pdx + Qdy = 0$ integratio ita perficietur. Quaeratur integrale $\int Pdx$ spectata y constante idque rursus differentietur spectata sola y variabili, unde prodeat Vdy; tum $Q - V$ erit functio ipsius y tantum, unde quaeratur $Y = \int (Q - V)dy$, eritque aequatio integralis $\int Pdx + Y = \text{Const.}$

COROLLARIUM 3

451. Vel quaeratur $\int Qdy$ spectata x constante, quod integrale rursus differentietur sumta x variabili, y autem constante, unde prodeat Udx; tum certe erit $P - U$ functio ipsius x tantum, unde quaeratur $X = \int (P - U)dx$, eritque aequatio integralis quaesita $\int Qdy + X = \text{Const.}$

COROLLARIUM 4

452. Ex rei natura patet perinde esse, utra via procedatur; necesse enim est ad eandem aequationem integralem perveniri, si quidem aequatio differentialis proposita per se fuerit integrabilis. Tum autem certe eveniet, ut priori casu $Q - V$ sit functio solius y, posteriori autem $P - U$ functio solius x.

SCHOLION

453. Haec methodus integrandi etiam tentari posset, antequam exploratum esset, num aequatio integrabilis existat; si enim vel in modo Corollarii 2 eveniret, ut $Q - V$ esset functio ipsius y tantum, vel in modo Corollarii 3, ut $P - U$ esset functio ipsius x tantum, hoc ipsum indicio foret aequationem esse per se integrabilem. Verum tamen praestat ante omnia scrutari, an aequatio integrabilis sit per se necne, seu an sit $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right)$, quoniam hoc examen sola differentiatione absolvitur. Exempla igitur aliquot aequationum per se integrabilium afferamus, quo non solum methodus integrandi, sed etiam insignes illae proprietates, quas commemoravimus, clarius intelligantur.

EXEMPLUM 1

454. *Aequationem per se integrabilem*

$$dx(\alpha x + \beta y + \gamma) + dy(\beta x + \delta y + \varepsilon) = 0$$

integrare.

Cum hic sit $P = \alpha x + \beta y + \gamma$ et $Q = \beta x + \delta y + \varepsilon$, erit $\left(\frac{dP}{dy}\right) = \beta$ et $\left(\frac{dQ}{dx}\right) = \beta$, qua aequalitate integrabilitas per se confirmatur. Quaeratur ergo per Corollarium 2 spectata y ut constante

$$\int P dx = \frac{1}{2}\alpha xx + \beta yx + \gamma x;$$

erit $V dy = \beta x dy$ et

$$(Q - V)dy = dy(\delta y + \varepsilon) = dY \quad \text{ideoque} \quad Y = \frac{1}{2}\delta yy + \varepsilon y,$$

unde integrale erit

$$\frac{1}{2}\alpha xx + \beta yx + \gamma x + \frac{1}{2}\delta yy + \varepsilon y = C.$$

Modo autem Corollarii 3 spectata x constante erit

$$\int Q dy = \beta xy + \frac{1}{2}\delta yy + \varepsilon y,$$

quae spectata y constante praebet $U dx = \beta y dx$ hincque

$$(P - U)dx = (\alpha x + \gamma)dx \quad \text{et} \quad X = \frac{1}{2}\alpha xx + \gamma x,$$

unde $\int Qdy + X = C$ integrale dat ut ante. Hinc simul etiam intelligitur esse

$$\int Pdx - \int Qdy = \frac{1}{2}\alpha xx + \gamma x - \frac{1}{2}\delta yy - \varepsilon y,$$

quae in duas functiones $X - Y$ sponte dispescitur.

EXEMPLUM 2

455. *Aequationem per se integrabilem*

$$\frac{dy}{y} = \frac{xdy - ydx}{y\sqrt{(xx+yy)}} \quad \textit{seu} \quad \frac{dx}{\sqrt{(xx+yy)}} + \frac{dy}{y}\left(1 - \frac{x}{\sqrt{(xx+yy)}}\right) = 0$$

integrare.

Cum hic sit $P = \frac{1}{\sqrt{(xx+yy)}}$ et $Q = \frac{1}{y} - \frac{x}{y\sqrt{(xx+yy)}}$, pro charactere integrabilitatis per se cognoscendo est $\left(\frac{dP}{dy}\right) = \frac{-y}{(xx+yy)^{\frac{3}{2}}}$ et $\left(\frac{dQ}{dx}\right) = \frac{-y}{(xx+yy)^{\frac{3}{2}}}$, qui bini valores utique sunt aequales. Iam pro integrali inveniendo utamur regula Corollarii 2 et habebimus

$$\int Pdx = l\left(x + \sqrt{(xx+yy)}\right) \quad \text{et} \quad Vdy = \frac{ydy}{(x + \sqrt{(xx+yy)})\sqrt{(xx+yy)}}$$

seu supra et infra per $\sqrt{(xx+yy)} - x$ multiplicando

$$V = \frac{\sqrt{(xx+yy)} - x}{y\sqrt{(xx+yy)}} = \frac{1}{y} - \frac{x}{y\sqrt{(xx+yy)}},$$

unde

$$Q - V = 0 \quad \text{et} \quad Y = \int (Q - V)dy = 0,$$

sicque integrale quaesitum

$$l\left(x + \sqrt{(xx+yy)}\right) = \text{Const.}$$

Per regulam Corollarii 3 habemus

$$\int Qdy = ly - x\int \frac{dy}{y\sqrt{(xx+yy)}},$$

at posito $y = \frac{1}{z}$ est

$$\int \frac{dy}{y\sqrt{(xx+yy)}} = -\int \frac{dz}{\sqrt{(xxzz+1)}} = -\frac{1}{x}l\left(xz + \sqrt{(xxzz+1)}\right),$$

ergo

$$\int Q dy = ly + l\frac{x + \sqrt{(xx+yy)}}{y} = l\left(x + \sqrt{(xx+yy)}\right),$$

unde

$$U dx = \frac{dx}{\sqrt{(xx+yy)}}, \quad \text{hinc} \quad (P - U) dx = 0.$$

EXEMPLUM 3

456. *Aequationem per se integrabilem*

$$(xx + yy - aa) dy + (aa + 2xy + xx) dx = 0$$

integrare.

Hic ergo est $P = aa + 2xy + xx$ et $Q = xx + yy - aa$, unde $\left(\frac{dP}{dy}\right) = 2x$ et $\left(\frac{dQ}{dx}\right) = 2x$, quae aequalitas integrabilitatem per se innuit. Tum vero est

$$\int P dx = aax + xxy + \frac{1}{3}x^3 \quad \text{et} \quad V dy = xxdy,$$

unde

$$(Q - V) dy = (yy - aa) dy \quad \text{et} \quad Y = \frac{1}{3}y^3 - aay.$$

Ergo integrale

$$aax + xxy + \frac{1}{3}x^3 + \frac{1}{3}y^3 - aay = \text{Const.}$$

Altero modo est

$$\int Q dy = xxy + \frac{1}{3}y^3 - aay \quad \text{hincque} \quad U dx = 2xydx,$$

ergo

$$(P - U) dx = (aa + xx) dx \quad \text{et} \quad X = aax + \frac{1}{3}x^3,$$

unde integrale oritur ut ante.

SCHOLION

457. In his exemplis licuit integrale $\int P dx$ actu exhibere indeque eius differentiale $V dy$ sumta sola y variabili assignare. Quodsi autem hoc integrale $\int P dx$ evolvi nequeat, haud liquet, quomodo inde differentiale $V dy$ elici possit, quandoquidem formula $\int P dx$ in se spectata constantem quamcunque, quae etiam y in se implicet, complectitur. Tum igitur quomodo procedendum sit, videamus.

Ponamus

$$Z = \int P dx + Y,$$

et cum quaeratur $\left(\frac{d.\int P dx}{dy}\right) = V$, ob $\int P dx = Z - Y$ erit $V = \left(\frac{dZ}{dy}\right) - \frac{dY}{dy}$. At est $\left(\frac{dZ}{dx}\right) = P$, ergo $\left(\frac{ddZ}{dx dy}\right) = \left(\frac{dP}{dy}\right) = \left(\frac{dV}{dx}\right)$ ob $\left(\frac{dZ}{dy}\right) = V + \frac{dY}{dy}$. Hinc erit

$$V = \int dx \left(\frac{dP}{dy}\right);$$

quare quantitas V invenitur per integrationem huius formulae $\int dx \left(\frac{dP}{dy}\right)$, in qua y ut constans spectatur, postquam in valore $\left(\frac{dP}{dy}\right)$ inveniendo sola y variabilis esset assumta. Verum cum hic denuo constans cum y implicetur, hinc illa functio Y, quam quaerimus, non determinatur. Ratio huius incommodi manifesto in ambiguitate integralium $\int P dx$ et $\int dx \left(\frac{dP}{dy}\right)$ est sita, dum utrumque functiones arbitrarias ipsius y recipit. Remedium ergo afferetur, si utrumque integrale certa quadam conditione determinetur. Ita quando integrale $\int P dx$ ita accipi ponimus, ut evanescat posito $x = f$, ubi quidem constantem f pro lubitu accipere licet, tum eadem lege alterum integrale $\int dx \left(\frac{dP}{dy}\right)$ capiatur. Quo facto erit $Q - \int dx \left(\frac{dP}{dy}\right)$ functio ipsius y tantum et aequationis $P dx + Q dy = 0$ integrale erit

$$\int P dx + \int dy \left(Q - \int dx \left(\frac{dP}{dy}\right)\right) = \text{Const.},$$

dummodo ambo integralia $\int P dx$ et $\int dx \left(\frac{dP}{dy}\right)$, in quibus y ut constans tractatur, ita determinentur, ut evanescant, dum in utroque ipsi x idem valor f tribuitur. Quare hinc istam colligimus regulam:

REGULA PRO INTEGRATIONE AEQUATIONIS PER SE INTEGRABILIS $P dx + Q dy = 0$ IN QUA $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right)$

458. *Quaerantur integralia* $\int P dx$ *et* $\int dx \left(\frac{dP}{dy}\right)$ *spectando* y *ut constantem ita, ut ambo evanescant, dum ipsi* x *certus quidam valor, puta* $x = f$, *tribuitur. Tum erit* $Q - \int dx \left(\frac{dP}{dy}\right)$ *functio ipsius* y *tantum, quae sit* $= Y$, *et integrale quaesitum erit* $\int P dx + \int Y dy =$ Const. *Vel, quod eodem redit, quaerantur integralia*

$\int Qdy$ *et* $\int dy\left(\frac{dQ}{dx}\right)$ *spectando* x *ut constantem ita, ut ambo evanescant, dum ipsi* y *certus quidam valor, puta* $y=g$, *tribuitur; tum* $P-\int dy\left(\frac{dQ}{dx}\right)$ *erit functio ipsius* x *tantum, qua posita* $=X$ *erit integrale quaesitum* $\int Qdy+\int Xdx=$ Const.

DEMONSTRATIO

Veritatem huius regulae ex praecedentibus perspicere licet, si cui forte precario assumsisse videamur ambas formulas $\int Pdx$ et $\int dx\left(\frac{dP}{dy}\right)$ eadem lege determinari debere, ut, dum ipsi x certus quidam valor, puta $x=f$, tribuitur, ambae evanescant. Sed ne forte quis putet alteram integrationem pari iure secundum aliam legem determinari posse, hanc demonstrationem addo. Prima quidem integratio ab arbitrio nostro pendet, quam ergo ita determinari assumamus, ut integrale $\int Pdx$ evanescat posito $x=f$; quo facto dico alterum integrale $\int dx\left(\frac{dP}{dy}\right)$ necessario per eandem conditionem determinari oportere. Sit enim $\int Pdx=Z$ eritque Z eiusmodi functio ipsarum x et y, quae evanescit posito $x=f$; habebit ergo factorem $f-x$ vel eius quampiam potestatem positivam $(f-x)^{\lambda}$, ita ut sit $Z=(f-x)^{\lambda}T$. Nunc quia $\int dx\left(\frac{dP}{dy}\right)$ exprimit valorem ipsius $\left(\frac{dZ}{dy}\right)$, erit $\int dx\left(\frac{dP}{dy}\right)=(f-x)^{\lambda}\left(\frac{dT}{dy}\right)$, ex quo manifestum est hoc integrale etiam evanescere posito $x=f$, ita ut huius integralis determinatio non amplius arbitrio nostro relinquatur. Hoc posito erit utique aequationis per se integrabilis $Pdx+Qdy=0$ integrale

$$\int Pdx+\int Ydy=\text{Const.}$$

existente

$$Y=Q-\int dx\left(\frac{dP}{dy}\right);$$

nam posito $\int Pdx=Z$, quatenus scilicet in hac integratione y pro constante habetur, habetur haec aequatio $Z+\int Ydy=$ Const., quam esse integrale quaesitum vel ex ipsa differentiatione patebit. Cum enim sit

$$dZ=Pdx+dy\left(\frac{dZ}{dy}\right)=Pdx+dy\int dx\left(\frac{dP}{dy}\right),$$

erit aequationis inventae differentiale

$$Pdx + dy\int dx\left(\frac{dP}{dy}\right) + Ydy = 0,$$

sed $Y = Q - \int dx\left(\frac{dP}{dy}\right)$, unde prodit $Pdx + Qdy = 0$, quae est ipsa aequatio differentialis proposita. Quod autem sit $Q - \int dx\left(\frac{dP}{dy}\right)$ functio ipsius y tantum, inde sequitur, quoniam aequatio differentialis per se est integrabilis.

THEOREMA

459. *Pro omni aequatione, quae per se non est integrabilis, semper datur quantitas, per quam ea multiplicata redditur integrabilis.*

DEMONSTRATIO

Sit $Pdx + Qdy = 0$ aequatio differentialis et concipiamus eius integrale completum, quod erit aequatio quaedam inter x et y, in quam constans quantitas arbitraria ingrediatur. Ex hac aequatione eruatur haec ipsa constans arbitraria, ut prodeat huiusmodi aequatio

$$\text{Const.} = \text{functioni cuidam ipsarum } x \text{ et } y,$$

quae differentiata praebeat

$$0 = Mdx + Ndy;$$

quae aequatio iam a constante illa arbitraria per integrationem ingressa est libera ideoque necesse est, ut haec aequatio differentialis conveniat cum proposita, alioquin integrale suppositum non esset verum. Oportet ergo, ut relatio inter dx et dy utrinque prodeat eadem, unde erit

$$\frac{P}{Q} = \frac{M}{N}$$

ideoque

$$M = LP \quad \text{et} \quad N = LQ.$$

Sed quia $Mdx + Ndy$ est verum differentiale ex differentiatione cuiuspiam functionis ipsarum x et y ortum, est $\left(\frac{dM}{dy}\right) = \left(\frac{dN}{dx}\right)$. Quare pro aequatione

$Pdx + Qdy = 0$ dabitur certo quidam multiplicator L, ut sit

$$\left(\frac{d.LP}{dy}\right) = \left(\frac{d.LQ}{dx}\right),$$

seu ut aequatio per L multiplicata fiat per se integrabilis.

COROLLARIUM 1

460. Pro omni ergo aequatione $Pdx + Qdy = 0$ datur eiusmodi functio L, ut sit $\left(\frac{d.LP}{dy}\right) = \left(\frac{d.LQ}{dx}\right)$ ideoque evolvendo

$$L\left(\frac{dP}{dy}\right) + P\left(\frac{dL}{dy}\right) = L\left(\frac{dQ}{dx}\right) + Q\left(\frac{dL}{dx}\right)$$

seu

$$L\left(\left(\frac{dP}{dy}\right) - \left(\frac{dQ}{dx}\right)\right) = Q\left(\frac{dL}{dx}\right) - P\left(\frac{dL}{dy}\right);$$

quae functio L si fuerit inventa, aequatio differentialis $LPdx + LQdy = 0$ per se erit integrabilis.

COROLLARIUM 2

461. In aequatione proposita loco Q tuto unitatem scribere licet, quia omnis aequatio hac forma $Pdx + dy = 0$ repraesentari potest. Hinc inventio multiplicatoris L, qui eam reddat per se integrabilem, pendet a resolutione huius aequationis

$$L\left(\frac{dP}{dy}\right) = \left(\frac{dL}{dx}\right) - P\left(\frac{dL}{dy}\right),$$

ubi notandum est esse

$$dL = dx\left(\frac{dL}{dx}\right) + dy\left(\frac{dL}{dy}\right).$$

SCHOLION

462. Quoniam hic quaeritur functio binarum variabilium x et y, quarum relatio mutua minime spectatur, quam involvit aequatio $Pdx + Qdy = 0$, haec investigatio in nostrum librum secundum incurrit, ubi huiusmodi functio ex data quadam differentialium relatione indagari debet. In hac enim investigatione non attendimus ad aequationem propositam, qua formula $Pdx + Qdy$

nihilo aequalis reddi debet, sed absolute quaeritur multiplicator L, per quem formula $Pdx + Qdy$ multiplicata abeat in verum differentiale cuiuspiam functionis finitae, quae sit Z, ita ut habeatur $dZ = LPdx + LQdy$. Quo multiplicatore L invento tum demum aequalitas $Pdx + Qdy = 0$ spectatur indeque concluditur functionem Z quantitati constanti aequari oportere. Cum igitur minime expectari queat, ut methodum tradamus huiusmodi multiplicatores pro quavis aequatione differentiali proposita inveniendi, eos casus percurramus, quibus talis multiplicator constat, undecunque sit repertus. Interim tamen ad pleniorem usum huius methodi notasse iuvabit, statim atque unum multiplicatorem pro quapiam aequatione differentiali cognoverimus, ex eo facile innumerabiles alios deduci posse, qui pariter aequationem propositam per se integrabilem reddant.

PROBLEMA 60

463. *Dato uno multiplicatore L, qui aequationem $Pdx + Qdy = 0$ per se integrabilem reddat, invenire innumerabiles alios multiplicatores, qui idem officium praestent.*

SOLUTIO

Cum ergo $L(Pdx + Qdy)$ sit differentiale verum cuiuspiam functionis Z, quaeratur per superiora praecepta haec functio Z, ita ut sit

$$L(Pdx + Qdy) = dZ,$$

et nunc manifestum est hanc formulam dZ integrationem etiam esse admissuram, si per functionem quamcunque ipsius Z, quam ita $\varphi:Z$ indicemus, multiplicetur. Cum igitur etiam integrabilis sit haec formula

$$(Pdx + Qdy)L\,\varphi:Z,$$

erit quoque $L\,\varphi:Z$ multiplicator aequationis propositae $Pdx + Qdy = 0$, qui eam reddat integrabilem. Quare invento uno multiplicatore L quaeratur per integrationem $Z = \int L(Pdx + Qdy)$ ac tum expressio $L\,\varphi:Z$, ubi pro $\varphi:Z$ functio quaecunque ipsius Z assumi potest, dabit infinitos alios multiplicatores idem officium praestantes.

SCHOLION

464. Tametsi sufficiat pro quavis aequatione differentiali unicum multiplicatorem cognovisse, tamen occurrunt casus, quibus perquam utile est plures, imo infinitos multiplicatores in promtu habere. Veluti si aequatio proposita in duas partes commode discerpatur huiusmodi

$$(Pdx + Qdy) + (Rdx + Sdy) = 0$$

atque omnes multiplicatores constent, quibus utraque pars seorsim $Pdx + Qdy$ et $Rdx + Sdy$ reddatur integrabilis, inde interdum communis multiplicator utramque integrabilem reddens concludi potest. Sit enim $L\varphi:Z$ expressio generalis pro omnibus multiplicatoribus formulae $Pdx + Qdy$ et $M\varphi:V$ expressio generalis pro omnibus multiplicatoribus formulae $Rdx + Sdy$, et quoniam $\varphi:Z$ et $\varphi:V$ functiones quascunque quantitatum Z et V denotant, si eas ita capere liceat, ut fiat $L\varphi:Z = M\varphi:V$, habebitur multiplicator idoneus pro aequatione

$$Pdx + Qdy + Rdx + Sdy = 0.$$

Intelligitur autem hoc iis tantum casibus praestari posse, quibus multiplicator pro tota aequatione etiam singulas eius partes seorsim sumtas integrabiles reddat. Quare cavendum est, ne huic methodo nimium tribuatur et, quando ea non succedit, aequatio pro irresolubili habeatur; evenire enim utique potest, ut tota aequatio habeat multiplicatorem, qui singulis eius partibus non conveniat. Ita proposita aequatione $Pdx + Qdy = 0$ multiplicator partem Pdx seorsim integrabilem reddens manifesto est $\frac{X}{P}$, denotante X functionem quamcunque ipsius x, et multiplicator partem alteram Qdy integrabilem reddens est $\frac{Y}{Q}$; etiamsi autem neutiquam fieri possit, ut sit $\frac{X}{P} = \frac{Y}{Q}$ seu $\frac{P}{Q} = \frac{X}{Y}$, nisi casibus per se obviis, tamen tota formula $Pdx + Qdy$ certo semper habet multiplicatorem, quo ea integrabilis reddatur.

EXEMPLUM 1

465. *Invenire omnes multiplicatores, quibus formula $\alpha y dx + \beta x dy$ integrabilis redditur.*

Primus multiplicator sponte se offert $\frac{1}{xy}$, qui praebet $\frac{\alpha dx}{x} + \frac{\beta dy}{y}$, cuius integrale est $\alpha lx + \beta ly = lx^\alpha y^\beta$. Huius ergo functio quaecunque $\varphi:x^\alpha y^\beta$ in $\frac{1}{xy}$ ducta dabit multiplicatorem idoneum, cuius itaque forma generalis est

$\frac{1}{xy}\varphi : x^\alpha y^\beta$. Functio enim quantitatis $x^\alpha y^\beta$ etiam est functio logarithmi eiusdem quantitatis. Nam si P fuerit functio ipsius p et Π functio ipsius P, etiam Π est functio ipsius p et vicissim.

COROLLARIUM

466. Si pro functione sumatur potestas quaecunque $x^{n\alpha}y^{n\beta}$, formula $\alpha y dx + \beta x dy$ integrabilis redditur, si multiplicetur per $x^{n\alpha-1}y^{n\beta-1}$, quo quidem casu integrale sponte patet; est enim $\frac{1}{n}x^{n\alpha}y^{n\beta}$.

EXEMPLUM 2

467. *Invenire omnes multiplicatores, qui hanc formulam $Xydx + dy$ integrabilem reddant.*

Primus multiplicator $\frac{1}{y}$ sponte se offert, unde, cum sit

$$\int\left(Xdx + \frac{dy}{y}\right) = \int Xdx + ly \quad \text{seu} \quad le^{\int Xdx}y,$$

omnes functiones huius quantitatis seu huius $e^{\int Xdx}y$ per y divisae dabunt multiplicatores idoneos. Unde expressio generalis pro omnibus multiplicatoribus erit $= \frac{1}{y}\varphi : e^{\int Xdx}y$.

COROLLARIUM

468. Pro formula ergo $Xydx + dy$ multiplicator quoque est $e^{\int Xdx}$, qui est functio ipsius x tantum; quo ergo cum etiam formula $\mathfrak{X}dx$ denotante $\mathfrak{X}$ functionem quamcunque ipsius x integrabilis reddatur, ille multiplicator etiam huic formulae $dy + Xydx + \mathfrak{X}dx$ conveniet.

PROBLEMA 61

469. *Proposita aequatione $dy + Xydx = \mathfrak{X}dx$, in qua X et $\mathfrak{X}$ sint functiones quaecunque ipsius x, invenire multiplicatorem idoneum eamque integrare.*

SOLUTIO

Cum alterum membrum $\mathfrak{X}dx$ per functionem quamcunque ipsius x multiplicatum fiat integrabile, dispiciatur, num etiam prius membrum $dy + Xydx$

per huiusmodi multiplicatorem integrabile reddi possit. Quod cum praestet multiplicator $e^{\int X dx}$, hoc adhibito habebitur aequatio integralis quaesita

$$e^{\int X dx} y = \int e^{\int X dx} \mathfrak{X} dx$$

sive

$$y = e^{-\int X dx} \int e^{\int X dx} \mathfrak{X} dx,$$

uti iam supra [§ 420] invenimus.

COROLLARIUM 1

470. Patet etiam, si loco y adsit functio quaecunque ipsius y, ut habeatur haec aequatio $dY + YXdx = \mathfrak{X} dx$, eam per multiplicatorem $e^{\int X dx}$ reddi integrabilem et integrale fore

$$e^{\int X dx} Y = \int e^{\int X dx} \mathfrak{X} dx.$$

COROLLARIUM 2

471. Quare etiam haec aequatio $dy + yXdx = y^n \mathfrak{X} dx$ quia per y^n divisa abit in

$$\frac{dy}{y^n} + \frac{Xdx}{y^{n-1}} = \mathfrak{X} dx,$$

ubi posito $\frac{1}{y^{n-1}} = Y$ ob $-\frac{(n-1)dy}{y^n} = dY$ seu $\frac{dy}{y^n} = -\frac{dY}{n-1}$ prodit

$$-\frac{dY}{n-1} + YXdx = \mathfrak{X} dx \quad \text{seu} \quad dY - (n-1) YXdx = -(n-1) \mathfrak{X} dx,$$

quae per multiplicatorem $e^{-(n-1)\int X dx}$ fit integrabilis: eiusque integrale erit

$$e^{-(n-1)\int X dx} Y = -(n-1) \int e^{-(n-1)\int X dx} \mathfrak{X} dx$$

sive [§ 429]

$$\frac{1}{y^{n-1}} = -(n-1) e^{(n-1)\int X dx} \int e^{-(n-1)\int X dx} \mathfrak{X} dx.$$

SCHOLION

472. Cum pro membro $dy + yXdx$ multiplicator generalis sit $\frac{1}{y} \varphi : e^{\int X dx} y$, sumta loco functionis potestate multiplicator idoneus erit $e^{m\int X dx} y^{m-1}$ inte-

grale praebens $\frac{1}{m}e^{m\int Xdx}y^m$. Efficiendum ergo est, ut etiam idem multiplicator alterum membrum $y^n\mathfrak{X}dx$ reddat integrabile; quod evenit sumendo $m-1=-n$ seu $m=1-n$, ex quo huius membri integrale fit $\int e^{m\int Xdx}\mathfrak{X}dx$, ita ut aequatio integralis quaesita obtineatur

$$\frac{1}{1-n}e^{(1-n)\int Xdx}y^{1-n}=\int e^{(1-n)\int Xdx}\mathfrak{X}dx,$$

quae cum modo inventa prorsus congruit.

PROBLEMA 62

473. *Proposita aequatione differentiali*

$$\alpha ydx+\beta xdy=x^m y^n(\gamma ydx+\delta xdy)$$

invenire multiplicatorem idoneum, qui eam integrabilem reddat, ipsumque integrale assignare.

SOLUTIO

Consideretur utrumque membrum seorsim; ac pro priori vidimus $\alpha ydx+\beta xdy$ omnes multiplicatores idoneos contineri in hac forma

$$\frac{1}{xy}\varphi : x^\alpha y^\beta.$$

Pro altera parte $x^m y^n(\gamma ydx+\delta xdy)$ primus multiplicator est

$$\frac{1}{x^{m+1}y^{n+1}},$$

quo prodit $\frac{\gamma dx}{x}+\frac{\delta dy}{y}$, cuius integrale est $lx^\gamma y^\delta$; ergo forma generalis pro eius multiplicatoribus est

$$\frac{1}{x^{m+1}y^{n+1}}\varphi : x^\gamma y^\delta.$$

Quo nunc hi duo multiplicatores pares reddantur, loco functionum sumantur potestates fiatque

$$x^{\mu\alpha-1}y^{\mu\beta-1}=x^{\nu\gamma-m-1}y^{\nu\delta-n-1},$$

unde statui oportet

$$\mu\alpha = \nu\gamma - m \quad \text{et} \quad \mu\beta = \nu\delta - n;$$

hincque colligitur

$$\mu = \frac{\gamma n - \delta m}{\alpha\delta - \beta\gamma} \quad \text{et} \quad \nu = \frac{\alpha n - \beta m}{\alpha\delta - \beta\gamma}.$$

Quocirca multiplicator erit

$$x^{\mu\alpha - 1} y^{\mu\beta - 1} = x^{\nu\gamma - m - 1} y^{\nu\delta - n - 1},$$

unde aequatio nostra induit hanc formam

$$x^{\mu\alpha - 1} y^{\mu\beta - 1}(\alpha y dx + \beta x dy) = x^{\nu\gamma - 1} y^{\nu\delta - 1}(\gamma y dx + \delta x dy),$$

ubi utrumque membrum per se est integrabile ideoque integrale quaesitum

$$\frac{1}{\mu} x^{\mu\alpha} y^{\mu\beta} = \frac{1}{\nu} x^{\nu\gamma} y^{\nu\delta} + \text{Const.},$$

quod convenit cum eo, quod capite praecedente [§ 431] est inventum.

COROLLARIUM 1

474. Posito ergo brevitatis gratia $\mu = \frac{\gamma n - \delta m}{\alpha\delta - \beta\gamma}$ et $\nu = \frac{\alpha n - \beta m}{\alpha\delta - \beta\gamma}$ aequationis differentialis

$$\alpha y dx + \beta x dy = x^m y^n (\gamma y dx + \delta x dy)$$

integrale completum est

$$\frac{1}{\mu} x^{\mu\alpha} y^{\mu\beta} = \frac{1}{\nu} x^{\nu\gamma} y^{\nu\delta} + \text{Const.}$$

COROLLARIUM 2

475. Si eveniat, ut sit $\mu = 0$ seu $\gamma n = \delta m$, integrale ad logarithmos reducetur eritque

$$l x^\alpha y^\beta = \frac{1}{\nu} x^{\nu\gamma} y^{\nu\delta} + \text{Const.};$$

sin autem $\nu = 0$ seu $\alpha n = \beta m$, erit integrale

$$\frac{1}{\mu} x^{\mu\alpha} y^{\mu\beta} = l x^\gamma y^\delta + \text{Const.}$$

SCHOLION

476. Hinc autem casus excipi videtur, quo $\alpha\delta = \beta\gamma$, quia tum ambo numeri μ et ν fiunt infiniti. Verum si $\delta = \frac{\beta\gamma}{\alpha}$, aequatio nostra hanc induit formam

$$\alpha y dx + \beta x dy = \frac{\gamma}{\alpha} x^m y^n (\alpha y dx + \beta x dy) \quad \text{seu} \quad (\alpha y dx + \beta x dy)(1 - \frac{\gamma}{\alpha} x^m y^n) = 0,$$

quae cum habeat duos factores, duplex solutio ex utroque seorsim ad nihilum reducto derivatur. Prior scilicet nascitur ex $\alpha y dx + \beta x dy = 0$, cuius integrale est $x^\alpha y^\beta = \text{Const.}$, alter vero factor per se dat aequationem finitam $1 - \frac{\gamma}{\alpha} x^m y^n = 0$, quarum solutionum utraque aeque satisfacit. Atque hoc in genere tenendum est de omnibus aequationibus differentialibus, quas in factores resolvere licet, ubi perinde atque in aequationibus finitis singuli factores praebent solutiones. Plerumque autem factores finiti statim, antequam integratio suscipitur, per divisionem tolli solent, quandoquidem non ex natura rei, sed per operationes institutas demum accessisse censentur, ita ut, perinde ac in Algebra saepe fieri solet, ad solutiones inutiles essent perducturi.

PROBLEMA 63

477. *Proposita aequatione differentiali homogenea multiplicatorem idoneum invenire, qui eam integrabilem reddat, indeque eius integrale eruere.*

SOLUTIO

Sit $Pdx + Qdy = 0$ aequatio proposita, in qua P et Q sint functiones homogeneae n dimensionum ipsarum x et y, ac quaeramus multiplicatorem L, qui sit etiam functio homogenea, cuius dimensionum numerus sit λ. Cum iam formula $L(Pdx + Qdy)$ sit integrabilis, erit integrale functio $\lambda + n + 1$ dimensionum ipsarum x et y, quae functio si ponatur Z, erit ex natura functionum homogenearum [§ 481]

$$LPx + LQy = (\lambda + n + 1)Z.$$

Quare si λ sumatur $= -n - 1$, quantitas $LPx + LQy$ erit vel $= 0$ vel constans, unde obtinemus $L = \frac{1}{Px + Qy}$, qui ergo est multiplicator idoneus pro nostra aequatione.

Idem quoque ex separatione variabilium colligitur; posito enim $y = ux$ fiet $P = x^n U$ et $Q = x^n V$ existentibus U et V functionibus u ipsius tantum et ob $dy = udx + xdu$ erit

$$Pdx + Qdy = x^n Udx + x^n Vudx + x^n Vxdu$$

seu

$$Pdx + Qdy = x^n(U + Vu)dx + x^{n+1} Vdu.$$

At haec formula per $x^{n+1}(U + Vu)$ divisa fit integrabilis ideoque et formula nostra $Pdx + Qdy$ divisa per $x^{n+1}(U + Vu) = Px + Qy$, restitutis valoribus $U = \frac{P}{x^n}$, $V = \frac{Q}{x^n}$ et $u = \frac{y}{x}$, fiet integrabilis; seu multiplicator idoneus est $\frac{1}{Px + Qy}$, unde haec aequatio $\frac{Pdx + Qdy}{Px + Qy} = 0$ semper per se est integrabilis.

Iam ad integrale ipsius inveniendum integretur formula $\int \frac{Pdx}{Px + Qy}$ spectando y ut constantem ac determinetur certa ratione, ut evanescat posito $x = f$. Tum posito brevitatis causa $\frac{P}{Px + Qy} = R$ sumatur valor $\left(\frac{dR}{dy}\right)$ et eadem lege quaeratur integrale $\int dx \left(\frac{dR}{dy}\right)$ spectando iterum y ut constantem. Tum erit $\frac{Q}{Px+Qy} - \int dx \left(\frac{dR}{dy}\right)$ functio ipsius y tantum seu $\frac{Q}{Px+Qy} - \int dx \left(\frac{dR}{dy}\right) = Y$ atque hinc erit integrale quaesitum

$$\int \frac{Pdx}{Px + Qy} + \int Ydy = \text{Const.}$$

COROLLARIUM 1

478. Cum ergo formula $\frac{Pdx + Qdy}{Px + Qy}$ sit per se integrabilis, si brevitatis gratia ponamus

$$\frac{P}{Px + Qy} = R \quad \text{et} \quad \frac{Q}{Px + Qy} = S,$$

necesse est sit $\left(\frac{dR}{dy}\right) = \left(\frac{dS}{dx}\right)$. At est

$$\left(\frac{dR}{dy}\right) = \left(Qy\left(\frac{dP}{dy}\right) - Py\left(\frac{dQ}{dy}\right) - PQ\right) : (Px + Qy)^2$$

et

$$\left(\frac{dS}{dx}\right) = \left(Px\left(\frac{dQ}{dx}\right) - Qx\left(\frac{dP}{dx}\right) - PQ\right) : (Px + Qy)^2.$$

Quamobrem habebitur

$$Qy\left(\frac{dP}{dy}\right) - Py\left(\frac{dQ}{dy}\right) = Px\left(\frac{dQ}{dx}\right) - Qx\left(\frac{dP}{dx}\right).$$

COROLLARIUM 2

479. Haec aequalitas etiam ex natura functionum homogenearum concluditur. Cum enim P et Q sint functiones n dimensionum ipsarum x et y, ob

$$dP = dx\left(\frac{dP}{dx}\right) + dy\left(\frac{dP}{dy}\right) \quad \text{et} \quad dQ = dx\left(\frac{dQ}{dx}\right) + dy\left(\frac{dQ}{dy}\right)$$

erit

$$nP = x\left(\frac{dP}{dx}\right) + y\left(\frac{dP}{dy}\right) \quad \text{et} \quad nQ = x\left(\frac{dQ}{dx}\right) + y\left(\frac{dQ}{dy}\right).$$

Aequalitas autem inventa est

$$Q\left(x\left(\frac{dP}{dx}\right) + y\left(\frac{dP}{dy}\right)\right) = P\left(x\left(\frac{dQ}{dx}\right) + y\left(\frac{dQ}{dy}\right)\right),$$

quae hinc abit in identicam $nPQ = nPQ$.

COROLLARIUM 3

480. Si aequatio homogenea $Pdx + Qdy = 0$ fuerit per se integrabilis et P et Q sint functiones -1 dimensionis, erit $Px + Qy$ numerus constans. Veluti cum $\frac{xdx + ydy}{xx + yy} = 0$ huiusmodi sit aequatio, si loco dx et dy scribantur x et y, prodit $\frac{xx + yy}{xx + yy} = 1$.

SCHOLION

481. In *Calculo Differentiali*[1]) ostendimus, si V fuerit functio homogenea n dimensionum ipsarum x et y ponaturque $dV = Pdx + Qdy$, fore

$$Px + Qy = nV.$$

Quare si $Pdx + Qdy$ fuerit formula integrabilis et P et Q functiones homogeneae $n-1$ dimensionum, integrale statim habetur; erit enim $V = \frac{1}{n}(Px + Qy)$

1) *Institutiones calculi differentialis,* partis prioris § 222. F. E.

neque ad hoc ulla integratione est opus. Interim tamen videmus hinc excipi oportere casum, quo $n=0$, uti fit in nostra aequatione per multiplicatorem integrabili reddita $\frac{Pdx+Qdy}{Px+Qy}=0$, ubi dx et dy multiplicantur per functiones -1 dimensionis; neque enim hic integrale sine integratione obtineri potest. Ratio autem huius exceptionis in hoc est sita, quod formulae integrabilis $Pdx+Qdy$, in qua P et Q sunt functiones homogeneae $n-1$ dimensionum, integrale tum tantum sit functio homogenea n dimensionum, quando n non est $=0$; hoc enim solo casu fieri potest, ut integrale non sit functio nullius dimensionis, quemadmodum fit in hac formula differentiali $\frac{xdx+ydy}{xx+yy}$, quippe cuius integrale est $\frac{1}{2}l(xx+yy)$. Quocirca, quod formula $\frac{Pdx+Qdy}{Px+Qy}$ sit integrabilis, hoc peculiari modo demonstravimus ex ratione separabilitatis deducto. Interim tamen sine ullo respectu, unde hoc cognoverimus, id in praesenti negotio maxime est notatu dignum omnes aequationes homogeneas $Pdx+Qdy=0$ per multiplicatorem $\frac{1}{Px+Qy}$ per se reddi integrabiles. Methodus igitur desideratur, cuius beneficio hunc multiplicatorem a priori invenire liceret; qua methodo sane maxima incrementa in Analysin importarentur. Quamdiu autem eousque pertingere non licet, plurimum intererit huiusmodi multiplicatores pro pluribus casibus probe notasse; quod cum iam in duobus aequationum generibus praestiterimus, pro reliquis aequationibus, quas supra integrare docuimus, multiplicatores investigemus; ipsa autem reductio ad separationem nobis hos multiplicatores patefaciet, uti in sequente problemate docebimus.

PROBLEMA 64

482. *Proposita aequatione differentiali, quam ad separationem variabilium reducere liceat, invenire multiplicatorem, per quem ea per se integrabilis reddatur.*

SOLUTIO

Sit $Pdx+Qdy=0$, quae certa quadam substitutione, dum loco x et y aliae binae variabiles t et u introducuntur, ad separationem accommodetur; ponamus ergo facta hac substitutione fieri $Pdx+Qdy=Rdt+Sdu$, nunc autem hanc formulam $Rdt+Sdu$, si per V dividatur, separari, ita ut in hac formula $\frac{Rdt+Sdu}{V}$ quantitas $\frac{R}{V}$ sit functio solius t et $\frac{S}{V}$ functio solius u. Cum igitur formula $\frac{Rdt+Sdu}{V}$ per se sit integrabilis, etiam integrabilis erit haec

$\frac{Pdx + Qdy}{V}$, quippe illi aequalis, siquidem in V variabiles x et y restituantur. Hinc ergo ex reductione ad separabilitatem aequationis $Pdx + Qdy = 0$ discimus multiplicatorem, quo ea integrabilis reddatur, esse $\frac{1}{V}$ sicque, quas aequationes ad separationem variabilium perducere licet, pro iisdem multiplicatorem, qui illas integrabiles reddat, assignare possumus.

COROLLARIUM 1

483. Methodus ergo per multiplicatores integrandi aequationes differentiales aeque late patet ac prior methodus ope separationis variabilium, propterea quod ipsa separatio pro quavis aequatione, ubi succedit, multiplicatorem suppeditat.

COROLLARIUM 2

484. Contra autem methodus per multiplicatores integrandi latius patet altera, si pro eiusmodi aequationibus multiplicatores assignare liceat, quas quomodo ad separationem perduci debeant, non constet.

SCHOLION

485. Etsi autem ex reductione ad separationem idoneum multiplicatorem elicere licet, tamen nondum intelligitur, quomodo cognito multiplicatore separatio variabilium institui debeat; quare etiam ob hanc rationem methodus per multiplicatores integrandi alteri longe praeferenda videtur. Quamvis enim hactenus ipsa separatio nos ad inventionem multiplicatorum perduxerit, nullum tamen est dubium, quin detur via multiplicatores inveniendi nullo respectu ad separationem habito, licet haec via etiamnum nobis sit incognita. Ea autem paullatim planior reddetur, si pro quam plurimis aequationibus multiplicatores idoneos cognoverimus, ex quo, quos adhuc ex separatione eruere licet, indagemus in subiunctis exemplis.

EXEMPLUM 1

486. *Proposita aequatione differentiali primi ordinis*

$$dx(\alpha x + \beta y + \gamma) + dy(\delta x + \varepsilon y + \zeta) = 0$$

pro ea multiplicatorem idoneum assignare.

Haec aequatio ad separationem praeparatur ponendo primo [§ 417]

$$\alpha x + \beta y + \gamma = r \quad \text{et} \quad \delta x + \varepsilon y + \zeta = s$$

ideoque

$$\alpha dx + \beta dy = dr \quad \text{et} \quad \delta dx + \varepsilon dy = ds,$$

unde oritur

$$dx = \frac{\varepsilon dr - \beta ds}{\alpha\varepsilon - \beta\delta} \quad \text{et} \quad dy = \frac{\alpha ds - \delta dr}{\alpha\varepsilon - \beta\delta},$$

hincque aequatio nostra omisso denominatore utpote constante erit

$$\varepsilon r dr - \beta r ds + \alpha s ds - \delta s dr = 0;$$

quae cum sit homogenea, per $\varepsilon rr - (\beta + \delta) rs + \alpha ss$ divisa fit integrabilis. Quod idem ex separatione colligitur; posito enim $r = su$ prodit

$$\varepsilon ssudu + \varepsilon suuds - \beta suds + \alpha sds - \delta ssdu - \delta suds = 0$$

seu

$$ssdu(\varepsilon u - \delta) + sds(\varepsilon uu - \beta u - \delta u + \alpha) = 0,$$

quae divisa per $ss(\varepsilon uu - \beta u - \delta u + \alpha)$ separatur. Quare multiplicator nostrae aequationis propositae est

$$\frac{1}{ss(\varepsilon uu - \beta u - \delta u + \alpha)} = \frac{1}{\varepsilon rr - \beta rs - \delta rs + \alpha ss} = \frac{1}{r(\varepsilon r - \beta s) + s(\alpha s - \delta r)},$$

qui restitutis valoribus fit

$$\frac{1}{(\alpha x + \beta y + \gamma)((\alpha\varepsilon - \beta\delta)x + \gamma\varepsilon - \beta\zeta) + (\delta x + \varepsilon y + \zeta)((\alpha\varepsilon - \beta\delta)y + \alpha\zeta - \gamma\delta)}$$

atque evolutione facta

$$1 : \left\{ \begin{matrix} (\alpha\varepsilon - \beta\delta)(\alpha xx + (\beta + \delta)xy + \varepsilon yy + \gamma x + \zeta y) + \alpha\zeta\zeta - (\beta + \delta)\gamma\zeta + \gamma\gamma\varepsilon \\ + (\alpha\gamma\varepsilon - (\beta - \delta)\alpha\zeta - \gamma\delta\delta)x + (\alpha\varepsilon\zeta + (\beta - \delta)\gamma\varepsilon - \beta\beta\zeta)y \end{matrix} \right\}$$

Quare per se integrabilis erit haec aequatio

$$\frac{dx(\alpha x + \beta y + \gamma) + dy(\delta x + \varepsilon y + \zeta)}{(\alpha\varepsilon - \beta\delta)(\alpha xx + (\beta + \delta)xy + \varepsilon yy + \gamma x + \zeta y) + Ax + By + C} = 0$$

existente

$$A = \alpha\gamma\varepsilon - (\beta - \delta)\alpha\zeta - \gamma\delta\delta,$$

$$B = \alpha\varepsilon\zeta + (\beta - \delta)\gamma\varepsilon - \beta\beta\zeta,$$

$$C = \alpha\zeta\zeta - (\beta + \delta)\gamma\zeta + \gamma\gamma\varepsilon.$$

COROLLARIUM

487. Etiamsi forte fiat $\alpha\varepsilon - \beta\delta = 0$, hic multiplicator non turbatur, cum tamen separatio non succedat hac quidem operatione. Sit enim $\alpha = ma$, $\beta = mb$, $\delta = na$, $\varepsilon = nb$, ut habeatur haec aequatio

$$dx(m(ax + by) + \gamma) + dy(n(ax + by) + \zeta) = 0;$$

ob

$$A = a(na - mb)(m\zeta - n\gamma), \quad B = b(na - mb)(m\zeta - n\gamma)$$

et

$$C = (m\zeta - n\gamma)(a\zeta - b\gamma)$$

omisso factore communi multiplicator est

$$\frac{1}{(na - mb)(ax + by) + a\zeta - b\gamma},$$

ita ut haec aequatio per se sit integrabilis

$$\frac{(ax + by)(mdx + ndy) + \gamma dx + \zeta dy}{(na - mb)(ax + by) + a\zeta - b\gamma} = 0.$$

EXEMPLUM 2

488. *Proposita aequatione differentiali*

$$ydx(c + nx) - dy(y + a + bx + nxx) = 0$$

multiplicatorem idoneum invenire.

Fiat substitutio [§ 433]

$$\frac{y(c + nx)}{y + a + bx + nxx} = u \quad \text{seu} \quad y = \frac{u(a + bx + nxx)}{c + nx - u},$$

ut contrahatur aequatio nostra in hanc formam

$$ydx(c + nx) - \frac{ydy(c + nx)}{u} = 0 \quad \text{seu} \quad \frac{y(c + nx)}{u}(udx - dy) = 0$$

vel

$$\frac{yy(c + nx)}{u}\left(\frac{dy}{y} - \frac{udx}{y}\right) = 0;$$

probe enim cavendum est, ne hic ullus factor omittatur. At facta substitutione reperitur

$$\frac{dy}{y}-\frac{udx}{y}=\frac{du}{u}+\frac{dx(b+2nx)}{a+bx+nxx}+\frac{du-ndx}{c+nx-u}-\frac{dx(c+nx-u)}{a+bx+nxx}$$
$$=\frac{du(c+nx)}{u(c+nx-u)}-\frac{dx(na+cc-bc+(b-2c)u+uu)}{(c+nx-u)(a+bx+nxx)}.$$

Unde aequatio nostra induet hanc formam

$$\frac{yy(c+nx)^2}{u(c+nx-u)}\left(\frac{du}{u}-\frac{dx(na+cc-bc+(b-2c)u+uu)}{(a+bx+nxx)(c+nx)}\right)=0,$$

quae ergo separabitur ducta in hunc multiplicatorem

$$\frac{u(c+nx-u)}{yy(c+nx)^2(na+cc-bc+(b-2c)u+uu)};$$

tum enim prodit

$$\frac{du}{u(na+cc-bc+(b-2c)u+uu)}-\frac{dx}{(a+bx+nxx)(c+nx)}=0.$$

Quo igitur multiplicatorem quaesitum consequamur, ibi loco u tantum opus est suum valorem restituere; tum autem reperitur multiplicator

$$\frac{a+bx+nxx}{n(a+bx+nxx)y^3+(a+bx+nxx)(2na-bc+n(b-2c)x)yy+(na+cc-bc)(a+bx+nxx)^2y},$$

qui reducitur ad hanc formam

$$\frac{1}{ny^3+(2na-bc)yy+n(b-2c)xyy+(na+cc-bc)(a+bx+nxx)y}.$$

EXEMPLUM 3

489. *Proposita aequatione differentiali*

$$\frac{ndx(1+yy)\sqrt{(1+yy)}}{\sqrt{(1+xx)}}+(x-y)dy=0$$

invenire multiplicatorem, qui eam integrabilem reddat.

Posuimus supra (§ 434)

$$y=\frac{x-u}{1+xu} \quad \text{seu} \quad u=\frac{x-y}{1+xy},$$

unde fit

$$x-y=\frac{u(1+xx)}{1+xu} \quad \text{et} \quad 1+yy=\frac{(1+xx)(1+uu)}{(1+xu)^2},$$

hincque nostra aequatio hanc induit formam

$$\frac{ndx(1+xx)(1+uu)^{\frac{3}{2}}}{(1+xu)^3}+\frac{udx(1+xx)(1+uu)-udu(1+xx)^2}{(1+xu)^3}=0,$$

quae primo multiplicata per $(1+xu)^3$, tum divisa per

$$(1+xx)^2(1+uu)(u+n\sqrt{(1+uu)})$$

separatur. Quare aequationis nostrae multiplicator erit

$$\frac{(1+xu)^3}{(1+xx)^2(1+uu)(u+n\sqrt{(1+uu)})},$$

qui primo ob $1+uu=\frac{(1+yy)(1+xu)^2}{1+xx}$ abit in $\frac{1+xu}{(1+xx)(1+yy)(u+n\sqrt{(1+uu)})}$. Nunc ob $u=\frac{x-y}{1+xy}$ est

$$\sqrt{(1+uu)}=\frac{\sqrt{(1+xx)(1+yy)}}{1+xy} \quad \text{et} \quad 1+xu=\frac{1+xx}{1+xy}$$

ideoque noster multiplicator colligitur

$$\frac{1}{(1+yy)(x-y+n\sqrt{(1+xx)(1+yy)})},$$

ita ut per se sit integrabilis haec aequatio

$$\frac{ndx(1+yy)\sqrt{(1+yy)}+(x-y)dy\sqrt{(1+xx)}}{(1+yy)(x-y+n\sqrt{(1+xx)(1+yy)})\sqrt{(1+xx)}}=0,$$

cuius integrationi non immoror, cum iam supra integrale exhibuerim.

EXEMPLUM 4

490. *Aliud exemplum memoratu dignum suppeditat haec aequatio*

$$ydx-xdy+ax^ny\,dy(x^n+b)^{\frac{1}{n}}=0.$$

Quae si hac forma repraesentetur

$$xdy-ydx+\frac{1}{b}x^{n+1}dy=\frac{1}{b}x^{n+1}dy+ax^ny\,dy(x^n+b)^{\frac{1}{n}},$$

evenit, ut utrumque integrabile existat, si ducatur in hunc multiplicatorem

$$\frac{y^{n-1}}{x^{n+1}+abx^ny(x^n+b)^{\frac{1}{n}}};$$

ad quem inveniendum ex separatione variabilium adhibeatur haec substitutio non adeo obvia

$$\frac{x}{(x^n+b)^{\frac{1}{n}}}=vy,$$

unde fit $x^n=\frac{bv^ny^n}{1-v^ny^n}$, et hinc aequatio

$$\frac{ydx-xdy}{(x^n+b)^{\frac{1}{n}}}+ax^nydy=0$$

abit in hanc

$$\frac{yydv+v^{n+1}y^{n+1}dy+abv^ny^{n+1}dy}{1-v^ny^n}=0,$$

quae multiplicata per $\frac{1-v^ny^n}{yyv^n(ab+v)}$ separatur

$$\frac{dv}{v^n(ab+v)}+y^{n-1}dy=0,$$

unde idem ille multiplicator colligitur.

EXEMPLUM 5

491. *Proposita aequatione differentiali*

$$dy+yydx-\frac{adx}{x^4}=0$$

invenire multiplicatorem, quo ea integrabilis reddatur.

Secundum § 436 ponatur $x=\frac{1}{t}$ et ob $dx=-\frac{dt}{tt}$ nostra formula erit $dy-\frac{yydt}{tt}+attdt$, in qua porro statuatur $y=t-ttz$, et prodibit

$$-tt(dz+zzdt-adt),$$

quae per $tt(zz-a)$ divisa separatur; ergo et nostra aequatio divisa per

$$tt(zz-a)=\frac{(t-y)^2-at^4}{tt}=(1-xy)^2-\frac{a}{xx}$$

fiet integrabilis, ex quo multiplicator erit

$$\frac{xx}{xx(1-xy)^2-a}$$

et aequatio per se integrabilis

$$\frac{x^4 dy + x^4 yydx - adx}{x^4(1-xy)^2 - axx} = 0.$$

Spectetur iam x ut constans eritque ex dy natum integrale

$$\frac{1}{2\sqrt{a}} l \frac{x(1-xy)+\sqrt{a}}{\sqrt{a}-x(1-xy)} + X;$$

pro quo ut valor ipsius X obtineatur, differentietur denuo ac prodibit

$$\frac{2xydx - dx}{xx(1-xy)^2 - a} + dX = \frac{x^4 yydx - adx}{x^4(1-xy)^2 - axx},$$

unde

$$dX = \frac{x^4 yydx - adx - 2x^3 ydx + xxdx}{x^4(1-xy)^2 - axx} = \frac{dx}{xx} \quad \text{et} \quad X = -\frac{1}{x} + C;$$

quare aequatio integralis completa erit

$$l \frac{\sqrt{a}+x(1-xy)}{\sqrt{a}-x(1-xy)} = \frac{2\sqrt{a}}{x} + C.$$

SCHOLION

492. En ergo plures casus aequationum differentialium, pro quibus multiplicatores novimus, ex quorum contemplatione haec insignis investigatio non parum adiuvari videtur. Quanquam autem adhuc longe absumus a certa methodo pro quovis casu multiplicatores idoneos inveniendi, hinc tamen formas aequationum colligere poterimus, ut per datos multiplicatores integrabiles reddantur; quod negotium cum in hac ardua doctrina maximam utilitatem allaturum videatur, in sequente capite aequationes investigabimus, quibus dati multiplicatores conveniant; exempla scilicet hic evoluta idoneas multiplicatorum formas nobis suppeditant, quibus nostram investigationem superstruere licebit.

CAPUT III

DE INVESTIGATIONE AEQUATIONUM DIFFERENTIALIUM QUAE PER MULTIPLICATORES DATAE FORMAE INTEGRABILES REDDANTUR

PROBLEMA 65

493. *Definire functiones P et Q ipsius x, ut aequatio differentialis*

$$Pydx + (y + Q)dy = 0$$

per multiplicatorem $\frac{1}{y^3 + Myy + Ny}$, *ubi M et N sunt functiones ipsius x, fiat integrabilis.*

SOLUTIO

Necesse igitur est, ut factoris ipsius dx, qui est $\frac{Py}{y^3 + Myy + Ny}$, differentiale ex variabilitate ipsius y natum aequale sit differentiali factoris ipsius dy, qui est $\frac{y + Q}{y^3 + Myy + Ny}$, dum sola x variabilis sumitur. Horum valorum aequalium neglecto denominatore communi aequalitas dat

$$-2Py^3 - PMy^2 = (y^3 + Myy + Ny)\frac{dQ}{dx} - (y + Q)\frac{(yydM + ydN)}{dx},$$

quae secundum potestates ipsius y ordinata praebet

$$\begin{aligned} 0 = 2Py^3dx &+ PMy^2dx \\ +y^3dQ &+ My^2dQ + NydQ \\ -y^3dM &- y^2dN \\ &- Qy^2dM - QydN \end{aligned}$$

unde singulis potestatibus seorsim ad nihilum perductis nanciscimur primo $NdQ - QdN = 0$ seu $\frac{dN}{N} = \frac{dQ}{Q}$, ex cuius integratione sequitur $N = \alpha Q$. Tum binae reliquae conditiones sunt

$$\text{I.} \quad 2Pdx + dQ - dM = 0$$

et

$$\text{II.} \quad PMdx + MdQ - \alpha dQ - QdM = 0,$$

unde $\text{I} \cdot M - \text{II} \cdot 2$ suppeditat

$$-MdQ - MdM + 2\alpha dQ + 2QdM = 0$$

seu

$$dQ + \frac{2QdM}{2\alpha - M} = \frac{MdM}{2\alpha - M},$$

quae per $(2\alpha - M)^2$ divisa et integrata dat

$$\frac{Q}{(2\alpha - M)^2} = \int \frac{MdM}{(2\alpha - M)^3} = -\int \frac{dM}{(2\alpha - M)^2} + 2\alpha \int \frac{dM}{(2\alpha - M)^3}$$

seu

$$\frac{Q}{(2\alpha - M)^2} = \frac{-1}{2\alpha - M} + \frac{\alpha}{(2\alpha - M)^2} + \beta = \frac{M - \alpha}{(2\alpha - M)^2} + \beta.$$

Erit ergo

$$Q = M - \alpha + \beta(2\alpha - M)^2$$

hincque

$$2Pdx = dM - dQ = +2\beta dM(2\alpha - M)$$

sicque pro M functionem quamcunque ipsius x sumere licet. Capiatur ergo $M = 2\alpha - X$; erit

$$Pdx = -\beta XdX \quad \text{et} \quad Q = \alpha - X + \beta XX$$

atque

$$N = \alpha\alpha - \alpha X + \alpha\beta XX.$$

Quocirca pro hac aequatione

$$-\beta yXdX + dy(\alpha - X + \beta XX + y) = 0$$

habemus hunc multiplicatorem

$$\frac{1}{y^3 + (2\alpha - X)yy + \alpha(\alpha - X + \beta XX)y},$$

quo ea integrabilis redditur.

COROLLARIUM 1

494. Tribuatur aequationi haec forma

$$dy(y + A + BV + CVV) - CyVdV = 0$$

eritque

$$\alpha = A, \quad X = -BV, \quad \beta XX = \beta BBVV = CVV,$$

ergo $\beta = \frac{C}{BB}$, unde multiplicator fiet

$$\frac{1}{y^3 + (2A + BV)yy + A(A + BV + CVV)y}.$$

COROLLARIUM 2

495. Si hic sumatur $V = a + x$, obtinebitur aequatio similis illi, quam supra § 488 integravimus, et multiplicator quoque cum eo, quem ibi dedimus, convenit. Hic autem multiplicator commodius hac forma exhibetur

$$\frac{1}{y(y + A)^2 + BVy(y + A) + ACVVy}.$$

COROLLARIUM 3

496. Si ponamus $y + A = z$, nostra aequatio erit

$$dz(z + BV + CVV) - C(z - A)VdV = 0,$$

cui convenit multiplicator

$$\frac{1}{(z - A)(zz + BVz + ACVV)},$$

ita ut per se integrabilis sit haec aequatio

$$\frac{dz(z + BV + CVV) - C(z - A)VdV}{(z - A)(zz + BVz + ACVV)} = 0.$$

SCHOLION

497. Quemadmodum hic aequationis $Pydx + (y + Q)dy = 0$ multiplicatorem assumsimus $= \frac{y^{-1}}{yy + My + N}$, ita generalius eius loco sumere poterimus

$\frac{y^{n-1}}{yy+My+N}$, ut haec aequatio

$$\frac{Py^n dx+(y^n+Qy^{n-1})dy}{yy+My+N}=0$$

per se debeat esse integrabilis, qua comparata cum forma $Rdx+Sdy=0$, ut sit $\left(\frac{dR}{dy}\right)=\left(\frac{dS}{dx}\right)$, habebimus

$$(n-2)Py^{n+1}+(n-1)PMy^n+nPNy^{n-1}$$
$$=(yy+My+N)y^{n-1}\frac{dQ}{dx}-(y^n+Qy^{n-1})\left(\frac{ydM}{dx}+\frac{dN}{dx}\right)$$

sive ordinata aequatione

$$\left.\begin{array}{lll}(n-2)Py^{n+1}dx & +(n-1)PMy^n dx & +nPNy^{n-1}dx\\ -y^{n+1}dQ- & My^n dQ- & Ny^{n-1}dQ\\ +y^{n+1}dM+ & y^n dN+ & Qy^{n-1}dN\\ + & Qy^n dM & \end{array}\right\}=0,$$

unde singulis membris ad nihilum reductis fit

$$\text{I.}\quad (n-2)Pdx=dQ-dM,$$
$$\text{II.}\quad (n-1)MPdx=MdQ-QdM-dN,$$
$$\text{III.}\quad nNPdx=NdQ-QdN.$$

Sit $Pdx=dV$ eritque ex prima $Q=A+M+(n-2)V$, quo valore in secunda substituto prodit

$$MdV+(n-2)VdM+AdM+dN=0,$$

et tertia fit

$$2NdV+(n-2)VdN+MdN-NdM+AdN=0,$$

unde eliminando dV reperitur

$$(n-2)V+A=\frac{MMdN-MNdM-2NdN}{2NdM-MdN}.$$

Verum si hinc vellemus V elidere, in aequationem differentio-differentialem illaberemur. Casus tamen, quo $n=2$, expediri potest.

EXEMPLUM

498. *Sit in evolutione huius casus $n = 2$, ut per se integrabilis esse debeat haec aequatio*

$$\frac{y(Pydx + (y+Q)dy)}{yy + My + N} = 0.$$

Ac primo esse oportet $Q = A + M$, tum vero

$$2ANdM - AMdN = M(MdN - NdM) - 2NdN,$$

quam ergo aequationem integrare debemus; quae cum in nulla iam tractatarum contineatur, videndum est, quomodo tractabilior reddi queat.

Ponatur ergo $M = Nu$, ut fiat

$$MdN - NdM = - NNdu$$

et

$$2NdM - MdN = 2NNdu + NudN,$$

hinc

$$2ANNdu + ANudN + N^3udu + 2NdN = 0$$

sive

$$\frac{2dN}{NN} + \frac{AudN}{NN} + \frac{2Adu}{N} + udu = 0;$$

statuatur porro $\frac{1}{N} = v$ seu $N = \frac{1}{v}$; habebitur

$$-2dv - Audv + 2Avdu + udu = 0$$

seu

$$dv - \frac{2Avdu}{2+Au} = \frac{udu}{2+Au},$$

ubi varibilis v unicam habet dimensionem, et hanc ob rem patet hanc aequationem integrabilem reddi, si dividatur per $(2 + Au)^2$, prodibitque

$$\frac{v}{(2+Au)^2} = \int \frac{udu}{(2+Au)^3} = \frac{C}{AA} - \frac{1+Au}{AA(2+Au)^2}$$

ideoque

$$v = \frac{C(2+Au)^2 - 1 - Au}{AA}.$$

Sumta ergo pro u functione quacunque ipsius x erit

$$N = \frac{AA}{C(2+Au)^2 - 1 - Au} \quad \text{et} \quad M = \frac{AAu}{C(2+Au)^2 - 1 - Au}$$

atque

$$Q = \frac{AC(2+Au)^2 - A}{C(2+Au)^2 - 1 - Au}.$$

Iam ex tertia aequatione adipiscimur

$$2NPdx = NdQ - QdN \quad \text{seu} \quad 2Pdx = Nd.\frac{Q}{N};$$

at

$$\frac{Q}{N} = \frac{C(2+Au)^2 - 1}{A}, \quad \text{unde} \quad d.\frac{Q}{N} = 2Cdu(2+Au),$$

ideoque

$$Pdx = \frac{AACdu(2+Au)}{C(2+Au)^2 - 1 - Au}.$$

Quocirca aequatio nostra per se integrabilis est

$$\frac{AACyydu(2+Au) + ydy(C(2+Au)^2y - (1+Au)y + AC(2+Au)^2 - A)}{C(2+Au)^2yy - (1+Au)yy + AAuy + AA} = 0,$$

quae posito $Au + 2 = t$ induet hanc formam

$$y \cdot \frac{ACytdt + ydy(Ctt - t + 1) + Ady(Ctt - 1)}{Cttyy - (t-1)yy + A(t-2)y + AA} = 0.$$

Hinc autem posito

$$A = \alpha, \quad C = \frac{\alpha\gamma}{\beta\beta} \quad \text{et} \quad t = -\frac{\beta x}{\alpha}$$

invenimus

$$y \cdot \frac{\alpha\gamma xydx + ydy(\alpha + \beta x + \gamma xx) - \alpha dy(\alpha - \gamma xx)}{(\alpha + \beta x + \gamma xx)yy - \alpha(2\alpha + \beta x)y + \alpha^3} = 0.$$

COROLLARIUM 1

499. Hoc igitur modo integrari potest haec aequatio

$$\alpha\gamma xydx + ydy(\alpha + \beta x + \gamma xx) - \alpha dy(\alpha - \gamma xx) = 0,$$

quae quomodo ad separationem reduci debeat, non statim patet. Est autem multiplicator idoneus

$$\frac{y}{(\alpha + \beta x + \gamma xx)yy - \alpha(2\alpha + \beta x)y + \alpha^3}.$$

COROLLARIUM 2

500. Hic multiplicator etiam hoc modo exprimi potest, ut eius denominator in factores resolvatur

$$(\alpha+\beta x+\gamma xx)y : \left\{ \begin{matrix} ((\alpha+\beta x+\gamma xx)y-\alpha(\alpha+\tfrac{1}{2}\beta x)+\alpha x\sqrt{(\tfrac{1}{4}\beta\beta-\alpha\gamma)}) \\ ((\alpha+\beta x+\gamma xx)y-\alpha(\alpha+\tfrac{1}{2}\beta x)-\alpha x\sqrt{(\tfrac{1}{4}\beta\beta-\alpha\gamma)}) \end{matrix} \right\}$$

COROLLARIUM 3

501. Si ergo ponamus

$$(\alpha+\beta x+\gamma xx)y-\alpha(\alpha+\frac{1}{2}\beta x)=\alpha z,$$

erit multiplicator

$$\frac{\alpha+\frac{1}{2}\beta x+z}{(z+x\sqrt{(\frac{1}{4}\beta\beta-\alpha\gamma)})(z-x\sqrt{(\frac{1}{4}\beta\beta-\alpha\gamma)})}.$$

At ob $y=\frac{\alpha\alpha+\frac{1}{2}\alpha\beta x+\alpha z}{\alpha+\beta x+\gamma xx}$ aequatio nostra erit

$$\gamma xydx+dy\left(z+\frac{1}{2}\beta x+\gamma xx\right)=0.$$

At est

$$dy=\frac{-\frac{1}{2}\alpha(\alpha\beta+4\alpha\gamma x+\beta\gamma xx)dx-\alpha zdx(\beta+2\gamma x)+\alpha dz(\alpha+\beta x+\gamma xx)}{(\alpha+\beta x+\gamma xx)^2};$$

hoc autem valore substituto prodit aequatio nimis complicata.

PROBLEMA 66

502. *Invenire aequationem differentialem huius formae*

$$yPdx+(Qy+R)dy=0,$$

in qua P, Q et R sint functiones ipsius x, ut ea integrabilis evadat per hunc multiplicatorem $\frac{y^m}{(1+Sy)^n}$, *ubi S est etiam functio ipsius x.*

SOLUTIO

Quia dx per $\frac{y^{m+1}P}{(1+Sy)^n}$ et dy per $\frac{Qy^{m+1}+Ry^m}{(1+Sy)^n}$ multiplicatur, oportet sit

$$(m+1)Py^m(1+Sy)-nPSy^{m+1}=\frac{(1+Sy)(y^{m+1}dQ+y^m dR)-nydS(Qy^{m+1}+Ry^m)}{dx},$$

qua evoluta aequatione erit

$$\left.\begin{aligned} (m+1)Py^m dx &+ (m+1-n)PSy^{m+1}dx - Sy^{m+2}dQ \\ -y^m dR &- \qquad\qquad y^{m+1}dQ + nQy^{m+2}dS \\ &- \qquad\qquad Sy^{m+1}dR \\ &+ \qquad\qquad nRy^{m+1}dS \end{aligned}\right\} = 0;$$

hinc fit

$$Pdx = \frac{dR}{m+1} \quad \text{et} \quad SdQ = nQdS$$

ideoque

$$Q = AS^n \quad \text{et} \quad dQ = nAS^{n-1}dS,$$

quibus in membro medio substitutis fit

$$\frac{m+1-n}{m+1}SdR - nAS^{n-1}dS - SdR + nRdS = 0$$

seu

$$-\frac{SdR}{m+1} - AS^{n-1}dS + RdS = 0$$

ideoque

$$dR - \frac{(m+1)RdS}{S} = -(m+1)AS^{n-2}dS,$$

quae per S^{m+1} divisa et integrata praebet

$$\frac{R}{S^{m+1}} = B - \frac{(m+1)AS^{n-m-2}}{n-m-2}.$$

Ponamus $A = (m+2-n)C$, ut sit

$$Q = (m+2-n)CS^n \quad \text{et} \quad R = BS^{m+1} + (m+1)CS^{n-1}$$

ideoque

$$Pdx = BS^m dS + (n-1)CS^{n-2}dS.$$

Quocirca habebimus hanc aequationem

$$ydS(BS^m + (n-1)CS^{n-2}) + dy((m+2-n)CS^n y + BS^{m+1} + (m+1)CS^{n-1}) = 0,$$

quae multiplicata per $\frac{y^m}{(1+Sy)^n}$ fit integrabilis, ubi pro S functionem quamcunque ipsius x capere licet.

COROLLARIUM 1

503. Integrari ergo poterit haec aequatio

$$ByS^m dS + BS^{m+1}dy + (n-1)CyS^{n-2}dS + (m+1)CS^{n-1}dy$$
$$+ (m+2-n)CS^n y dy = 0,$$

quae sponte resolvitur in has duas partes

$$BS^m(ydS + Sdy) + CS^{n-2}((n-1)ydS + (m+1)Sdy + (m+2-n)S^2 y dy) = 0,$$

quarum utraque seorsim per $\frac{y^m}{(1+Sy)^n}$ multiplicata fit integrabilis.

COROLLARIUM 2

504. Prior pars $BS^m(ydS + Sdy)$ integrabilis redditur per hunc multiplicatorem $\frac{1}{S^m}\varphi:Sy$; est enim haec formula $B(ydS + Sdy)\varphi:Sy$ per se integrabilis. Unde pro hac parte multiplicator erit $S^{\lambda-m}y^\lambda(1+Sy)^\mu$, qui utique continet assumtum $\frac{y^m}{(1+Sy)^n}$, si quidem capiatur $\lambda = m$ et $\mu = -n$. Est vero

$$\int \frac{y^m}{(1+Sy)^n}\cdot BS^m(ydS + Sdy) = B\int \frac{v^m dv}{(1+v)^n}$$

posito $Sy = v$.

COROLLARIUM 3

505. Pro altera parte, quae posito $S = \frac{1}{v}$ abit in

$$\frac{C}{v^n}(-(n-1)ydv + (m+1)vdy + (m+2-n)ydy),$$

habebimus

$$-\frac{(n-1)Cy}{v^n}\left(dv - \frac{(m+1)vdy}{(n-1)y} - \frac{(m+2-n)dy}{n-1}\right)$$

$$= -\frac{(n-1)Cy^{\frac{m+n}{n-1}}}{v^n}\left(y^{\frac{-m-1}{n-1}}dv - \frac{m+1}{n-1}y^{\frac{-m-n}{n-1}}vdy - \frac{m+2-n}{n-1}y^{\frac{-m-1}{n-1}}dy\right)$$

$$= -\frac{(n-1)Cy^{\frac{m+n}{n-1}}}{v^n}d.\left(y^{\frac{-m-1}{n-1}}v + y^{\frac{n-m-2}{n-1}}\right).$$

Ideoque haec altera pars ita repraesentabitur

$$-(n-1)CS^n y^{\frac{m+n}{n-1}}\, d.\frac{1+Sy}{y^{\frac{m+1}{n-1}}S}.$$

Multiplicator ergo hanc partem integrabilem reddens erit in genere

$$\frac{1}{S^n y^{\frac{m+n}{n-1}}}\varphi : \frac{1+Sy}{Sy^{\frac{m+1}{n-1}}}.$$

COROLLARIUM 4

506. Pro altera ergo parte multiplicator erit $\frac{(1+Sy)^\mu}{S^{n+\mu} y^{\frac{m+n+\mu(m+1)}{n-1}}}$, quo haec pars fit

$$-(n-1)C\frac{(1+Sy)^\mu}{S^\mu y^{\frac{\mu(m+1)}{n-1}}} d.\frac{1+Sy}{y^{\frac{m+1}{n-1}}S},$$

cuius integrale est $-\frac{(n-1)CZ^{\mu+1}}{\mu+1}$ posito $Z=\frac{1+Sy}{y^{\frac{m+1}{n-1}}S}$.

COROLLARIUM 5

507. Iam multiplicator pro prima parte $S^{\lambda-m}y^\lambda(1+Sy)^\mu$ congruens reddetur cum multiplicatore alterius partis modo exhibito, si sumatur $\lambda=m$ et $\mu=-n$, unde resultat multiplicator communis $\frac{y^m}{(1+Sy)^n}$, hincque posito

$$Sy=v \quad \text{et} \quad \frac{1+Sy}{y^{\frac{m+1}{n-1}}S}=z$$

nostrae aequationis integrale erit

$$B\int\frac{v^m dv}{(1+v)^n}+Cz^{1-n}=D \quad \text{sive} \quad B\int\frac{v^m dv}{(1+v)^n}+\frac{CS^{n-1}y^{m+1}}{(1+Sy)^{n-1}}=D.$$

SCHOLION

508. Aequatio ergo, quam hoc problemate integrare didicimus, per principia iam supra stabilita tractari potest, dum pro binis eius partibus seorsim multiplicatores quaeruntur iique inter se congruentes redduntur, cuius methodi hic insignem usum declaravimus.

Possemus etiam multiplicatori hanc formam dare $\frac{y^m}{(1+Sy+Tyy)^n}$, ita ut haec aequatio

$$\frac{y^m(yPdx+(Qy+R)dy)}{(1+Sy+Tyy)^n}=0$$

per se debeat esse integrabilis, et calculo ut ante instituto inveniemus

$$\left.\begin{array}{llll} (m+1)Py^m dx & +(m+1-n)PSy^{m+1}dx & +(m+1-2n)PTy^{m+2}dx & -\ Ty^{m+3}dQ \\ -y^m dR- & y^{m+1}dQ- & Sy^{m+2}dQ & +nQy^{m+3}dT \\ - & Sy^{m+1}dR- & Ty^{m+2}dR & \\ + & nRy^{m+1}dS+ & nQy^{m+2}dS & \\ & + & nRy^{m+2}dT & \end{array}\right\}=0,$$

unde ex ultimo membro $-TdQ+nQdT=0$ concludimus $Q=AT^n$ et ex primo $Pdx=\frac{dR}{m+1}$, qui valores in binis mediis substituti praebent

$$RdS-\frac{SdR}{m+1}-AT^{n-1}dT=0$$

et

$$RdT-\frac{2TdR}{m+1}+AT^n dS-AST^{n-1}dT=0,$$

quarum illa fit integrabilis per se, si $m=-2$, haec vero integrari potest, si $m=2n-1$; fit enim

$$RdT-\frac{TdR}{n}+AT^{n-1}(TdS-SdT)=0$$

seu

$$\frac{nRdT-TdR}{nT^{n+1}}+\frac{A(TdS-SdT)}{TT}=0,$$

cuius integrale est

$$\frac{-R}{nT^n}+\frac{AS}{T}=\frac{-B}{n}$$

hincque $R=BT^n+nAT^{n-1}S$. Praeterea vero notari meretur casus $m=-1$, quem cum illis in subiunctis exemplis evolvamus.

EXEMPLUM 1

509. *Definire hanc aequationem $yPdx+(Qy+R)dy=0$, ut multiplicata per $\frac{1}{y(1+Sy+Tyy)^n}$ fiat per se integrabilis.*

Ob $m = -1$ habemus statim $dR = 0$ ideoque $R = C$; tum est ut ante $Q = AT^n$ et $dQ = nAT^{n-1}dT$, unde binae reliquae determinationes erunt

$$-PSdx - AT^{n-1}dT + CdS = 0,$$
$$-2PTdx - AST^{n-1}dT + AT^n dS + CdT = 0;$$

hinc eliminando Pdx prodit

$$ASST^{n-1}dT - 2AT^n dT - AT^n SdS + 2CTdS - CSdT = 0.$$

Statuatur hic $SS = Tv$, ut fiat

$$2TdS - SdT = TS\left(\frac{2dS}{S} - \frac{dT}{T}\right) = \frac{TSdv}{v} = \frac{Tdv\sqrt{T}}{\sqrt{v}},$$

eritque

$$\frac{1}{2}AT^n vdT - 2AT^n dT - \frac{1}{2}AT^{n+1}dv + \frac{CTdv\sqrt{T}}{\sqrt{v}} = 0$$

seu hoc modo

$$-\frac{1}{2}AT^{n+2}d.\frac{v-4}{T} + \frac{CTdv\sqrt{T}}{\sqrt{v}} = 0,$$

cuius prior pars integrabilis redditur per multiplicatorem $\frac{1}{T^{n+2}}\varphi : \frac{v-4}{T}$, posterior vero per $\frac{1}{T\sqrt{T}}\varphi : v$, unde communis multiplicator erit $\frac{1}{T(v-4)^{n+\frac{1}{2}}\sqrt{T}}$, hincque aequatio elicitur integralis haec

$$\frac{AT^{n-\frac{1}{2}}}{(2n-1)(v-4)^{n-\frac{1}{2}}} + C\int\frac{dv}{(v-4)^{n+\frac{1}{2}}\sqrt{v}} = D,$$

unde T definitur per v; tum vero est

$$S = \sqrt{Tv}, \quad R = C, \quad Q = AT^n \quad \text{et} \quad Pdx = \frac{CdS - AT^{n-1}dT}{S}.$$

COROLLARIUM 1

510. Casu, quo est $n = \frac{1}{2}$, ob $\frac{1}{0}z^0 = lz$ habetur

$$\frac{1}{2}Al\frac{T}{v-4} + C\int\frac{dv}{(v-4)\sqrt{v}} = \frac{1}{2}D \quad \text{seu} \quad \frac{1}{2}Al\frac{T}{v-4} - \frac{1}{2}Cl\frac{\sqrt{v}+2}{\sqrt{v}-2} = \frac{1}{2}D,$$

unde posito $v = 4uu$ et $C = \lambda A$ erit

$$l\frac{T}{1-uu} - \lambda l\frac{1+u}{1-u} = \text{Const.}$$

seu

$$T = E(1-uu)\left(\frac{1+u}{1-u}\right)^{\lambda}.$$

Hinc porro

$$S = 2u\sqrt{T} = 2u\left(\frac{1+u}{1-u}\right)^{\frac{\lambda}{2}}\sqrt{E(1-uu)} \quad \text{et} \quad R = C = \lambda A;$$

tum

$$Q = A\left(\frac{1+u}{1-u}\right)^{\frac{\lambda}{2}}\sqrt{E(1-uu)}$$

atque

$$Pdx = \frac{\lambda A du}{u} + \frac{\lambda A dT}{2T} - \frac{A dT}{2Tu}.$$

At est

$$\frac{dT}{T} = \frac{-2udu + 2\lambda du}{1-uu}.$$

Ergo

$$Pdx = \frac{Adu(1+\lambda\lambda - 2\lambda u)}{1-uu}.$$

Quocirca pro hac aequatione

$$\frac{Aydu(1+\lambda\lambda-2\lambda u)}{1-uu} + Ady\left(\lambda + y\left(\frac{1+u}{1-u}\right)^{\frac{\lambda}{2}}\sqrt{E(1-uu)}\right) = 0$$

multiplicator erit

$$\frac{1}{y\sqrt{\left(1+2uy\left(\frac{1+u}{1-u}\right)^{\frac{\lambda}{2}}\sqrt{E(1-uu)} + Eyy(1-uu)\left(\frac{1+u}{1-u}\right)^{\lambda}\right)}}.$$

COROLLARIUM 2

511. Casu, quo $n = -\frac{1}{2}$, habemus

$$-\frac{A(v-4)}{2T} + 2C\sqrt{v} = -2D \quad \text{seu} \quad T = \frac{A(v-4)}{4D+4C\sqrt{v}}.$$

Ponamus $v = 4uu$, ut sit $T = \frac{A(uu-1)}{D+2Cu}$; tum fit

$$S = 2u\sqrt{T} = 2u\sqrt{\frac{A(uu-1)}{D+2Cu}}, \quad R = C, \quad Q = \sqrt{\frac{A(D+2Cu)}{uu-1}}$$

et

$$Pdx = \frac{Cdu}{u} + \frac{CdT}{2T} - \frac{AdT}{2TTu} = \frac{Cdu}{u} + \frac{du(C+Du+Cuu)(Cu^3-3Cu-D)}{u(uu-1)^2(D+2Cu)}\text{[1]},$$

unde tam aequatio quam multiplicator definitur.

EXEMPLUM 2

512. *Definire aequationem* $yPdx + (Qy+R)dy = 0$, *ut multiplicata per* $\frac{1}{y^2(1+Sy+Tyy)^n}$ *fiat per se integrabilis.*

Ob $m = -2$ ex superioribus [§ 508] habemus

$$RS = \frac{A}{n}T^n + B \quad \text{seu} \quad R = \frac{AT^n}{nS} + \frac{B}{S},$$

qui valor in altera aequatione substitutus praebet

$$\frac{(2n+1)AT^n dT}{nS} - \frac{2AT^{n+1}dS}{nSS} + AT^n dS - AST^{n-1}dT + \frac{BdT}{S} - \frac{2BTdS}{SS} = 0,$$

quae in has tres partes distinguatur

$$\frac{AS}{nT^n}\left(\frac{(2n+1)T^{2n}dT}{S^2} - \frac{2T^{2n+1}dS}{S^3}\right) + AT^{n+1}\left(\frac{dS}{T} - \frac{SdT}{TT}\right)$$
$$+ BS\left(\frac{dT}{SS} - \frac{2TdS}{S^3}\right) = 0$$

seu

$$\frac{AS}{nT^n}d.\frac{T^{2n+1}}{SS} + AT^{n+1}d.\frac{S}{T} + BSd.\frac{T}{SS} = 0.$$

Statuamus ad abbreviandum

$$\frac{T^{2n+1}}{SS} = p, \quad \frac{S}{T} = q \quad \text{et} \quad \frac{T}{SS} = r;$$

1) In editione principe haec formula caret expressione $\frac{Cdu}{u}$. F. E.

$=\frac{1}{qqr}$, hinc $p=\frac{1}{q^{4n}r^{2n-1}}$ nostraque aequatio ita se habebit

$$\frac{'p}{\sqrt{r}}dq+\frac{B}{qr}dr=0 \quad \text{seu} \quad \frac{A\sqrt{r}}{n\sqrt{p}}dp+\frac{A\sqrt{p}}{q\sqrt{r}}dq+Bdr=0.$$

Quas tres partes seorsim consideremus ac prima fit integrabilis multiplicata per $\frac{\sqrt{p}}{\sqrt{r}}\varphi:p$, secunda vero per $\frac{q\sqrt{r}}{\sqrt{p}}\varphi:q$, tertia vero per $\varphi:r$. Ut bini primi conveniant, ponatur

$$\frac{\sqrt{p}}{\sqrt{r}}p^{\lambda}=\frac{q\sqrt{r}}{\sqrt{p}}q^{\mu}$$

seu $p^{\lambda+1}=q^{\mu+1}r$, hinc $p=q^{\frac{\mu+1}{\lambda+1}}r^{\frac{1}{\lambda+1}}=q^{-4n}r^{-2n+1}$. Fit ergo $\lambda+1=-\frac{1}{2n-1}$ et $\mu+1=-4n(\lambda+1)=\frac{4n}{2n-1}$ sicque

$$\mu=\frac{2n+1}{2n-1} \quad \text{et} \quad \lambda=-\frac{2n}{2n-1}.$$

Multiplicetur ergo aequatio per $\frac{q^{\frac{4n}{2n-1}}\sqrt{r}}{\sqrt{p}}=q^{2n+\frac{4n}{2n-1}}r^{n}$ [1]) ac prodibit

$$\frac{A}{n}p^{\lambda}dp+Aq^{\mu}dq+Bq^{2n+\frac{4n}{2n-1}}r^{n}dr=0$$

seu

$$Ad.\left(\frac{p^{\lambda+1}}{n(\lambda+1)}+\frac{q^{\mu+1}}{\mu+1}\right)+Bq^{\frac{4nn+2n}{2n-1}}r^{n}dr=0$$

vel

$$\frac{(2n-1)A}{4n}d.q^{\frac{4n}{2n-1}}(1-4r)+Bq^{\frac{4nn+2n}{2n-1}}r^{n}dr=0.$$

Multiplicetur per $q^{\frac{4\nu n}{2n-1}}(1-4r)^{\nu}$, ut prodeat

$$\frac{(2n-1)A}{4n}q^{\frac{4\nu n}{2n-1}}(1-4r)^{\nu}d.q^{\frac{4n}{2n-1}}(1-4r)+Bq^{\frac{4nn+2n+4\nu n}{2n-1}}r^{n}dr(1-4r)^{\nu}=0.$$

1) Editio princeps loco r^{n} habet r^{n+1} et in formula sequenti $\frac{4n}{2n+1}$ loco $\frac{4n}{2n-1}$; quamobrem omnes formulae sequentes § 512—514 corrigendae erant. In editione tertia (Petropoli 1824) formulae § 512 correctae sunt, non autem formulae § 513 et 514. F. E.

Fiat ergo $4\nu + 4n + 2 = 0$ seu $\nu = -n - \frac{1}{2}$ et ambo membra integrari poterunt eritque

$$\frac{(2n-1)A}{4n(\nu+1)} q^{\frac{4n(\nu+1)}{2n-1}} (1-4r)^{\nu+1} + B\int r^n dr (1-4r)^\nu = \text{Const.};$$

at est $\nu + 1 = -n + \frac{1}{2} = -\frac{2n-1}{2}$ sicque habebitur

$$-\frac{A}{2n} q^{-2n} (1-4r)^{-\frac{2n-1}{2}} + B\int \frac{r^n dr}{(1-4r)^{\frac{2n+1}{2}}} = \text{Const.}$$

Dabitur ergo q per r eritque $S = \frac{1}{qr}$, $T = \frac{S}{q}$, tum

$$R = \frac{AT^n}{nS} + \frac{B}{S}, \quad Q = AT^n \quad \text{et} \quad Pdx = -dR.$$

COROLLARIUM 1

513. Si sit $n = -\frac{1}{2}$, erit

$$Aq(1-4r) + 2B\sqrt{r} = C$$

seu

$$q = \frac{C - 2B\sqrt{r}}{A(1-4r)};$$

hincque

$$S = \frac{A(1-4r)}{r(C-2B\sqrt{r})}, \quad T = \frac{A^2(1-4r)^2}{r(C-2B\sqrt{r})^2}, \quad Q = \frac{\sqrt{r}(C-2B\sqrt{r})}{1-4r}$$

et

$$R = \frac{Q+nB}{nS} = \frac{B-2Q}{S} = \frac{r(B-2C\sqrt{r})(C-2B\sqrt{r})}{A(1-4r)^2}$$

seu

$$R = \frac{BCr - 2(B^2+C^2)r\sqrt{r} + 4BCr^2}{A(1-4r)^2}.$$

COROLLARIUM 2

514. Ponamus eodem casu $r = \frac{1}{4}uu$[1]; erit

1) Editio princeps: $r = uu$. Ad formulas autem correctas (vide notam p. 324) substitutio $r = \frac{1}{4}uu$ magis apta videtur. F. E.

$$S = \frac{4A(1-uu)}{uu(C-Bu)}, \qquad T = \frac{4AA(1-uu)^2}{uu(C-Bu)^2},$$

$$Q = \frac{u(C-Bu)}{2(1-uu)}, \qquad R = \frac{uu(B-Cu)(C-Bu)}{4A(1-uu)^2}$$

hincque

$$Pdx = \frac{(B^2+C^2)(3uu+u^4)-2BC(u+3u^3)}{4A(1-uu)^3}du$$

eritque aequatio $yPdx + (Qy+R)dy = 0$ integrabilis, si multiplicetur per

$$\frac{\sqrt{(1+Sy+Tyy)}}{yy} = \frac{1}{yy}\sqrt{\left(1+\frac{4A(1-uu)y}{uu(C-Bu)}+\frac{4AA(1-uu)^2yy}{uu(C-Bu)^2}\right)}.$$

EXEMPLUM 3

515. *Definire aequationem $yPdx + (Qy+R)dy = 0$, quae multiplicata per $\frac{y^{2n-1}}{(1+Sy+Tyy)^n}$ fiat per se integrabilis.*

Hic est $m = 2n-1$, $Q = AT^n$ et $Pdx = \frac{dR}{2n}$, tum vero ex superioribus [§ 508] $R = nAT^{n-1}S + BT^n$ ac superest aequatio

$$RdS - \frac{SdR}{2n} - AT^{n-1}dT = 0,$$

quae loco R substituto valore invento abit in

$$(2n-1)AT^{n-1}SdS - (n-1)AT^{n-2}SSdT - 2AT^{n-1}dT$$
$$+ 2BT^n dS - BT^{n-1}SdT = 0$$

seu

$$(2n-1)ATSdS - (n-1)ASSdT - 2ATdT$$
$$+ 2BTTdS - BTSdT = 0.$$

Prius membrum posito $SS = u$ abit in

$$\left(n-\frac{1}{2}\right)ATdu - (n-1)AudT - 2ATdT$$

seu

$$\left(n-\frac{1}{2}\right)AT\left(du - \frac{(n-1)udT}{\left(n-\frac{1}{2}\right)T} - \frac{2dT}{n-\frac{1}{2}}\right)$$

sive

$$\frac{1}{2}(2n-1)AT^{\frac{4n-3}{2n-1}}\left(\frac{du}{T^{\frac{2n-2}{2n-1}}}-\frac{2(n-1)udT}{(2n-1)T^{\frac{4n-3}{2n-1}}}-\frac{4dT}{(2n-1)T^{\frac{2n-2}{2n-1}}}\right)$$

$$=\frac{1}{2}(2n-1)AT^{\frac{4n-3}{2n-1}}d.\left(\frac{u}{T^{\frac{2n-2}{2n-1}}}-4T^{\frac{1}{2n-1}}\right)$$

vel

$$\frac{1}{2}(2n-1)AT^{\frac{4n-3}{2n-1}}d.\,T^{\frac{1}{2n-1}}\left(\frac{SS}{T}-4\right)+\frac{BT^3}{S}d.\,\frac{SS}{T}=0$$

seu

$$(2n-1)AT^{\frac{-1}{2n-1}}d.\,T^{\frac{1}{2n-1}}\left(\frac{SS}{T}-4\right)+\frac{2BT}{S}d.\,\frac{SS}{T}=0.$$

Ponatur

$$\frac{SS}{T}=p \quad \text{et} \quad T^{\frac{1}{2n-1}}\left(\frac{SS}{T}-4\right)=q=T^{\frac{1}{2n-1}}(p-4),$$

ut sit $T^{\frac{1}{2n-1}}=\frac{q}{p-4}$, unde

$$T=\frac{q^{2n-1}}{(p-4)^{2n-1}} \quad \text{et} \quad S=\sqrt{\frac{pq^{2n-1}}{(p-4)^{2n-1}}}.$$

Ergo

$$\frac{(2n-1)A(p-4)dq}{q}+\frac{2B\sqrt{q^{2n-1}}}{\sqrt{p(p-4)^{2n-1}}}dp=0$$

sive

$$\frac{(2n-1)Adq}{q^{n+\frac{1}{2}}}+\frac{2Bdp:\sqrt{p}}{(p-4)^{n+\frac{1}{2}}}=0,$$

quae integrata praebet

$$\frac{-2A}{q^{n-\frac{1}{2}}}+2B\int\frac{dp:\sqrt{p}}{(p-4)^{n+\frac{1}{2}}}=2C,$$

et facto $\frac{p}{p-4}=vv$ seu $p=\frac{4vv}{vv-1}$ fiet

$$\frac{-2A}{q^{n-\frac{1}{2}}}-\frac{B}{4^{n-1}}\int dv(vv-1)^{n-1}=C.$$

SCHOLION

516. Haec fusius non prosequor, quia ista exempla eum in finem potissimum attuli, ut methodus supra tradita aequationes differentiales tractandi

exerceretur; in his enim exemplis casus non parum difficiles se obtulerunt, quos ita per partes resolvere licuit, ut pro singulis multiplicatores idonei quaererentur ex iisque multiplicator communis definiretur; nunc igitur alia aequationum genera, quae per multiplicatores integrabiles reddi queant, investigemus.

PROBLEMA 67

517. *Ipsius x functiones P, Q, R, S definire, ut haec aequatio*

$$(Py + Q)dx + ydy = 0$$

per hunc multiplicatorem $(yy + Ry + S)^n$ integrabilis reddatur.

SOLUTIO

Necesse igitur est sit

$$\left(\frac{d.(Py+Q)(yy+Ry+S)^n}{dy}\right) = \left(\frac{d.y(yy+Ry+S)^n}{dx}\right),$$

unde colligitur per $(yy + Ry + S)^{n-1}$ dividendo

$$P(yy + Ry + S) + n(Py + Q)(2y + R) = \frac{ny(ydR + dS)}{dx}$$

seu

$$\left.\begin{array}{lll}(2n+1)\,Pyydx & +(n+1)PRydx + & PSdx \\ -nyydR + & 2nQydx + nQRdx & \\ - & nydS & \end{array}\right\} = 0.$$

Hinc ergo concluditur

$$Pdx = \frac{ndR}{2n+1} \quad \text{et} \quad \frac{(n+1)RdR}{2n+1} + 2Qdx - dS = 0, \quad \frac{SdR}{2n+1} + QRdx = 0$$

porroque

$$Qdx = \frac{-SdR}{(2n+1)R} = \frac{-(n+1)RdR}{2(2n+1)} + \frac{dS}{2},$$

ergo

$$dS + \frac{2SdR}{(2n+1)R} = \frac{(n+1)RdR}{2n+1},$$

quae per $R^{\frac{2}{2n+1}}$ multiplicata et integrata dat

$$R^{\frac{2}{2n+1}} S = C + \frac{1}{4} R^{\frac{4n+4}{2n+1}},$$

hincque

$$S = \frac{1}{4} RR + C R^{\frac{-2}{2n+1}}$$

atque

$$Qdx = \frac{-RdR}{4(2n+1)} - \frac{C}{2n+1} R^{\frac{-2n-3}{2n+1}} dR \quad \text{et} \quad Pdx = \frac{ndR}{2n+1},$$

unde aequationem obtinemus

$$\left(ny - \frac{1}{4} R - C R^{\frac{-2n-3}{2n+1}}\right) dR + (2n+1) y dy = 0,$$

quae integrabilis redditur per hunc multiplicatorem

$$\left(yy + Ry + \frac{1}{4} RR + C R^{\frac{-2}{2n+1}}\right)^n.$$

COROLLARIUM 1

518. Casu, quo $n = -\frac{1}{2}$, fit $dR = 0$ et $R = A$ et reliquae aequationes sunt

$$(n+1) APdx + 2nQdx - ndS = 0 \quad \text{et} \quad PSdx + nAQdx = 0.$$

Ergo

$$Pdx = \frac{AQdx}{2S} = \frac{2Qdx - dS}{A} \quad \text{ideoque} \quad (AA - 4S) Qdx = -2SdS$$

seu

$$Qdx = \frac{-2SdS}{AA - 4S} \quad \text{et} \quad Pdx = \frac{-AdS}{AA - 4S}$$

sicque haec aequatio

$$\frac{(Ay + 2S) dS}{4S - AA} + ydy = 0$$

integrabilis redditur per hunc multiplicatorem $\frac{1}{\sqrt{(yy + Ay + S)}}$.

COROLLARIUM 2

519. Si hic ponamus $A = 2a$ et $S = x$, haec aequatio

$$\frac{(ay + x)\,dx + 2y\,dy\,(x - aa)}{(x - aa)\sqrt{(yy + 2ay + x)}} = 0$$

per se est integrabilis, unde integrale inveniri potest huius aequationis

$$x\,dx + ay\,dx + 2xy\,dy - 2aay\,dy = 0,$$

quae divisa per $(x - aa)\sqrt{(yy + 2ay + x)}$ fit integrabilis.

COROLLARIUM 3

520. Ad integrale inveniendum sumatur primo x constans et partis

$$\frac{2y\,dy}{\sqrt{(yy + 2ay + x)}}$$

integrale est

$$2\sqrt{(yy + 2ay + x)} + 2al\left(a + y - \sqrt{(yy + 2ay + x)}\right) + X;$$

cuius differentiale sumto y constante

$$\frac{dx}{\sqrt{(yy + 2ay + x)}} - \frac{a\,dx : \sqrt{(yy + 2ay + x)}}{a + y - \sqrt{(yy + 2ay + x)}} + dX$$

si alteri aequationis parti

$$\frac{(ay + x)\,dx}{(x - aa)\sqrt{(yy + 2ay + x)}}$$

aequetur, reperitur $dX = \frac{a\,dx}{aa - x}$ et $X = -al(aa - x)$. Ex quo integrale completum erit

$$\sqrt{(yy + 2ay + x)} + al\frac{a + y - \sqrt{(yy + 2ay + x)}}{\sqrt{(aa - x)}} = C.$$

COROLLARIUM 4

521. Memoratu dignus est etiam casus $n = -1$, qui scripto a loco $C + \frac{1}{4}$ praebet hanc aequationem $(y + aR)\,dR + y\,dy = 0$, quae divisa per $yy + Ry + aRR$ fit integrabilis; haec autem aequatio est homogenea.

SCHOLION

522. Potest etiam aequationis $(Py + Q)dx + ydy = 0$ multiplicator statui $(y + R)^m(y + S)^n$ fierique debet

$$\left(\frac{d.(Py+Q)(y+R)^m(y+S)^n}{dy}\right) = \left(\frac{d.y(y+R)^m(y+S)^n}{dx}\right),$$

unde reperitur

$$Pdx(y + R)(y + S) + mdx(Py + Q)(y + S) + ndx(Py + Q)(y + R)$$
$$= my(y + S)dR + ny(y + R)dS,$$

quae evolvitur in

$$\left.\begin{array}{r} (m+n+1)Pyydx + (n+1)PRydx + PRSdx \\ -myydR + (m+1)PSydx + mQSdx \\ -nyydS + (m+n)Qydx + nQRdx \\ - mSydR \\ - nRydS \end{array}\right\} = 0,$$

unde colligitur

$$Pdx = \frac{mdR + ndS}{m+n+1} \quad \text{et} \quad Qdx = \frac{-PRSdx}{mS+nR} = \frac{-RS(mdR+ndS)}{(m+n+1)(mS+nR)}$$

hincque

$$\frac{(mdR+ndS)((n+1)R+(m+1)S)}{m+n+1} - \frac{(m+n)RS(mdR+ndS)}{(m+n+1)(mS+nR)} - mSdR - nRdS = 0$$

seu

$$m(n+1)RdR - mnRdS + n(m+1)SdS - mnSdR$$
$$-\frac{m(m+n)RSdR + n(m+n)RSdS}{mS+nR} = 0,$$

quae reducitur ad hanc formam

$$(n+1)RRdR + (m-n-1)RSdR - mSSdR$$
$$+ (m+1)SSdS + (n-m-1)RSdS - nRRdS = 0;$$

quae cum sit homogenea, dividatur per

$$(n+1)R^3 + (m-2n-1)R^2S + (n-2m-1)RSS + (m+1)S^3$$

seu per
$$(R-S)^2((n+1)R+(m+1)S),$$
ut fiat integrabilis.

At ipsa illa aequatio per $R-S$ divisa erit
$$(n+1)RdR+mSdR-nRdS-(m+1)SdS=0.$$
Dividatur per $(R-S)((n+1)R+(m+1)S)$ et resolvatur in fractiones partiales
$$\frac{dR}{m+n+2}\left(\frac{m+n+1}{R-S}+\frac{n+1}{(n+1)R+(m+1)S}\right)$$
$$+\frac{dS}{m+n+2}\left(\frac{m+n+1}{S-R}+\frac{m+1}{(n+1)R+(m+1)S}\right)=0$$
seu
$$\frac{(m+n+1)(dR-dS)}{R-S}+\frac{(n+1)dR+(m+1)dS}{(n+1)R+(m+1)S}=0,$$
unde integrando obtinemus
$$(R-S)^{m+n+1}((n+1)R+(m+1)S)=C.$$
Sit $R-S=u$; erit
$$(n+1)R+(m+1)S=\frac{C}{u^{m+n+1}}$$
hincque
$$R=\frac{(m+1)u}{m+n+2}+\frac{a}{u^{m+n+1}} \quad \text{et} \quad S=-\frac{(n+1)u}{m+n+2}+\frac{a}{u^{m+n+1}},$$
tum vero
$$Pdx=\frac{(m-n)du}{m+n+2}-\frac{(m+n)adu}{u^{m+n+2}}$$
et
$$Qdx=\frac{du}{u}\left(\frac{a}{u^{m+n+1}}+\frac{(m+1)u}{m+n+2}\right)\left(\frac{a}{u^{m+n+1}}-\frac{(n+1)u}{m+n+2}\right).$$

COROLLARIUM 1

523. Hinc ergo integrari potest ista aequatio
$$ydy+ydu\left(\frac{m-n}{m+n+2}-\frac{(m+n)a}{u^{m+n+2}}\right)$$
$$+\frac{du}{u}\left(\frac{aa}{u^{2m+2n+2}}+\frac{(m-n)a}{(m+n+2)u^{m+n}}-\frac{(m+1)(n+1)uu}{(m+n+2)^2}\right)=0,$$

quippe quae per se fit integrabilis, si multiplicetur per

$$\left(y+\frac{a}{u^{m+n+1}}+\frac{(m+1)u}{m+n+2}\right)^m\left(y+\frac{a}{u^{m+n+1}}-\frac{(n+1)u}{m+n+2}\right)^n.$$

COROLLARIUM 2

524. Sit $m=n$ et aequatio nostra erit

$$ydy-\frac{2naydu}{u^{2n+2}}+\frac{aadu}{u^{4n+3}}-\frac{1}{4}udu=0,$$

cuius multiplicator est $\left(\left(y+\frac{a}{u^{2n+1}}\right)^2-\frac{1}{4}uu\right)^n$. Quare si ponamus $y=z-\frac{a}{u^{2n+1}}$, aequatio prodit

$$zdz-\frac{adz}{u^{2n+1}}+\frac{azdu}{u^{2n+2}}-\frac{1}{4}udu=0,$$

quae integrabilis fit multiplicata per $(zz-\frac{1}{4}uu)^n$. Vel ponatur $z=\frac{1}{2}y$ et $a=\frac{1}{2}b$; erit

$$ydy-udu-\frac{bdy}{u^{2n+1}}+\frac{bydu}{u^{2n+2}}=0$$

et multiplicator $(yy-uu)^n$.

COROLLARIUM 3

525. Si $m=-n$, prodit haec aequatio

$$ydy-nydu+\frac{aadu}{u^3}+\frac{1}{4}(nn-1)udu-\frac{nadu}{u}=0,$$

quae integrabilis redditur multiplicata per

$$\left(y+\frac{a}{u}-\frac{1}{2}(n+1)u\right)^n\left(y+\frac{a}{u}-\frac{1}{2}(n-1)u\right)^{-n}.$$

Posito autem $y+\frac{a}{u}=z$ prodit haec aequatio

$$zdz-nzdu+\frac{1}{4}(nn-1)udu-\frac{adz}{u}+\frac{azdu}{uu}=0,$$

quam integrabilem reddit hic multiplicator

$$\left(z-\frac{1}{2}(n+1)u\right)^n\left(z-\frac{1}{2}(n-1)u\right)^{-n}.$$

COROLLARIUM 4

526. Ponamus hic $z = uv$ et habebitur ista aequatio

$$uuvdv + udu\left(vv - nv + \frac{1}{4}(nn-1)\right) = adv;$$

quae si multiplicetur per $\left(\frac{v-\frac{1}{2}(n+1)}{v-\frac{1}{2}(n-1)}\right)^n$, utrumque membrum fiet integrabile. Posito enim

$$\frac{v-\frac{1}{2}(n+1)}{v-\frac{1}{2}(n-1)} = s \quad \text{seu} \quad v = \frac{n+1-(n-1)s}{2(1-s)}$$

oritur

$$\frac{s^{n+1}udu}{(1-s)^2} + \frac{n+1-(n-1)s}{2(1-s)^3}uus^n ds = \frac{as^n ds}{(1-s)^2},$$

cuius integrale est

$$\frac{s^{n+1}uu}{2(1-s)^2} = a\int\frac{s^n ds}{(1-s)^2}.$$

SCHOLION

527. Quo nostram aequationem in genere concinniorem reddamus, ponamus $m = -\lambda - 1 + \mu$ et $n = -\lambda - 1 - \mu$, ut sit $m + n + 2 = -2\lambda$, fietque aequatio

$$ydy - ydu\left(\frac{\mu}{\lambda} - 2(\lambda+1)au^{2\lambda}\right) + udu\left(\frac{\mu\mu - \lambda\lambda}{4\lambda\lambda} - \frac{\mu}{\lambda}au^{2\lambda} + aau^{4\lambda}\right) = 0,$$

quae per hunc multiplicatorem integrabilis redditur

$$\left(y + au^{2\lambda+1} - \frac{(\mu-\lambda)u}{2\lambda}\right)^{\mu-\lambda-1}\left(y + au^{2\lambda+1} - \frac{(\mu+\lambda)u}{2\lambda}\right)^{-\mu-\lambda-1}.$$

Ponatur $y + au^{2\lambda+1} = uz$ et orietur haec aequatio

$$uzdz - au^{2\lambda+1}dz + du\left(zz - \frac{\mu}{\lambda}z + \frac{\mu\mu-\lambda\lambda}{4\lambda\lambda}\right) = 0,$$

cui respondet multiplicator

$$u^{-2\lambda-1}\left(z + \frac{\lambda-\mu}{2\lambda}\right)^{\mu-\lambda-1}\left(z - \frac{\lambda+\mu}{2\lambda}\right)^{-\mu-\lambda-1}.$$

Reperitur autem integrale

$$C = a\int dz\left(z+\frac{\lambda-\mu}{2\lambda}\right)^{\mu-\lambda-1}\left(z-\frac{\lambda+\mu}{2\lambda}\right)^{-\mu-\lambda-1} + \frac{1}{2\lambda u^{2\lambda}}\left(z+\frac{\lambda-\mu}{2\lambda}\right)^{\mu-\lambda}\left(z-\frac{\lambda+\mu}{2\lambda}\right)^{-\mu-\lambda},$$

quod ergo convenit huic aequationi differentiali

$$zdz + \frac{du}{u}\left(z+\frac{\lambda-\mu}{2\lambda}\right)\left(z-\frac{\lambda+\mu}{2\lambda}\right) = au^{2\lambda}dz.$$

PROBLEMA 68

528. *Ipsius x functiones P, Q, R et X definire, ut haec aequatio*

$$dy + yydx + Xdx = 0$$

integrabilis reddatur per hunc multiplicatorem $\frac{1}{Pyy+Qy+R}$.

SOLUTIO

Debet ergo esse

$$\frac{1}{dy}d.\frac{yy+X}{Pyy+Qy+R} = \frac{1}{dx}d.\frac{1}{Pyy+Qy+R}$$

hincque

$$2y(Pyy+Qy+R)-(yy+X)(2Py+Q) = \frac{-yydP-ydQ-dR}{dx};$$

ergo fieri debet

$$\left.\begin{array}{l} Qyydx + \quad 2Rydx - QXdx \\ +\, yydP - 2PXydx + \qquad dR \\ \qquad +\qquad ydQ \end{array}\right\} = 0.$$

Quare habetur

$$Q = -\frac{dP}{dx} = \frac{dR}{Xdx} \quad \text{et} \quad X = -\frac{dR}{dP}.$$

Sumto ergo dx constante est $dQ = -\frac{ddP}{dx}$, unde fieri oportet

$$2Rdx + \frac{2PdRdx}{dP} - \frac{ddP}{dx} = 0 \quad \text{seu} \quad RdP + PdR = \frac{dPddP}{2dx^2},$$

cuius integratio praebet

$$PR = \frac{dP^2}{4dx^2} + C,$$

hinc

$$R = \frac{dP^2}{4Pdx^2} + \frac{C}{P}, \quad \text{tum} \quad Q = -\frac{dP}{dx} \quad \text{et} \quad X = \frac{C}{PP} + \frac{dP^2}{4PPdx^2} - \frac{ddP}{2Pdx^2}.$$

Ponamus $P = SS$, ut S sit functio quaecunque ipsius x, obtinebimusque

$$P = SS, \quad Q = -\frac{2SdS}{dx}, \quad R = \frac{C}{SS} + \frac{dS^2}{dx^2} \quad \text{et} \quad X = \frac{C}{S^4} - \frac{ddS}{Sdx^2},$$

quibus sumtis valoribus per se integrabilis erit haec aequatio

$$\frac{dy + yydx + Xdx}{Pyy + Qy + R} = 0.$$

SCHOLION

529. Haec solutio commodius institui poterit, si multiplicatori tribuatur haec forma $\frac{P}{yy + 2Qy + R}$, ut fieri debeat

$$\frac{1}{dy} d. \frac{P(yy + X)}{yy + 2Qy + R} = \frac{1}{dx} d. \frac{P}{yy + 2Qy + R},$$

unde oritur

$$\left.\begin{array}{l} 2PQyydx + 2PRydx - 2PQXdx \\ -yydP - 2PXydx - \qquad RdP \\ - \quad 2QydP + \qquad PdR \\ + \quad 2PydQ \end{array}\right\} = 0,$$

ubi ex singulis commode definitur $\frac{dP}{P}$, scilicet

$$\frac{dP}{P} = 2Qdx = \frac{Rdx - Xdx + dQ}{Q} = \frac{dR - 2QXdx}{R}.$$

Hinc colligitur $2Q(R + X)dx = dR$, unde nunc ipsum elementum dx definiamus $dx = \frac{dR}{2Q(R + X)}$; quo valore substituto adipiscimur

$$\frac{QdR}{R + X} = \frac{(R - X)dR}{2Q(R + X)} + dQ$$

seu

$$2QQdR = RdR - XdR + 2QRdQ + 2QXdQ,$$

unde colligimus

$$X = \frac{2QQdR - 2QRdQ - RdR}{2QdQ - dR} \quad \text{et} \quad R + X = \frac{2(QQ - R)dR}{2QdQ - dR},$$

hinc

$$dx = \frac{2QdQ - dR}{4Q(QQ - R)} \quad \text{atque} \quad \frac{dP}{P} = \frac{2QdQ - dR}{2(QQ - R)}$$

ideoque $P = A\sqrt{(QQ - R)}$.

Fiat $QQ - R = S$ ac reperietur

$$dx = \frac{dS}{4QS}, \quad X = \frac{4QSdQ}{dS} - QQ - S, \quad R = QQ - S$$

atque $P = A\sqrt{S}$. Quocirca habebimus hanc aequationem

$$dy + \frac{yydS}{4QS} + dQ - \frac{(QQ + S)dS}{4QS} = 0,$$

quae integrabilis redditur per hunc multiplicatorem

$$\frac{\sqrt{S}}{yy + 2Qy + QQ - S} = \frac{\sqrt{S}}{(y + Q)^2 - S}.$$

Ad eius integrale inveniendum sumantur Q et S constantes prodibitque

$$\int \frac{dy\sqrt{S}}{(y + Q)^2 - S} = \frac{1}{2} l \frac{y + Q - \sqrt{S}}{y + Q + \sqrt{S}} + V$$

existente V certa functione ipsius S vel Q. Iam differentietur haec forma sumta y constante proditque

$$\frac{dQ\sqrt{S} - \frac{(Q + y)dS}{2\sqrt{S}}}{(y + Q)^2 - S} + dV = \frac{yydS + 4QSdQ - QQdS - SdS}{4Q((y + Q)^2 - S)\sqrt{S}}$$

ideoque

$$dV = \frac{yydS + 2QydS + QQdS - SdS}{4Q((y + Q)^2 - S)\sqrt{S}} = \frac{dS}{4Q\sqrt{S}}.$$

Ex quo aequationis nostrae integrale est

$$\frac{1}{2} l \frac{y + Q - \sqrt{S}}{y + Q + \sqrt{S}} + \frac{1}{4}\int \frac{dS}{Q\sqrt{S}} = C.$$

COROLLARIUM 1

530. Singularis est casus, quo $R = QQ$; fit enim

$$\frac{dP}{P} = 2Qdx = \frac{QQdx - Xdx + dQ}{Q} = \frac{2dQ - 2Xdx}{Q},$$

unde has duas aequationes elicimus

$$QQdx + Xdx - dQ = 0 \quad \text{et} \quad QQdx + Xdx - dQ = 0;$$

quae cum inter se conveniant, erit

$$Xdx = dQ - QQdx \quad \text{et} \quad lP = 2\int Qdx.$$

COROLLARIUM 2

531. Sumto ergo Q negativo, ut habeamus hanc aequationem

$$dy + yydx - dQ - QQdx = 0,$$

haec integrabilis redditur per hunc multiplicatorem $\frac{e^{-2\int Qdx}}{(y-Q)^2}$. Et integrale erit

$$\frac{-1}{y-Q}e^{-2\int Qdx} + V = \text{Const.},$$

ubi V est functio ipsius x, ad quam definiendam differentietur sumta y constante

$$\frac{-dQ}{(y-Q)^2}e^{-2\int Qdx} + \frac{2Qdx}{y-Q}e^{-2\int Qdx} + dV = \frac{yydx - dQ - QQdx}{(y-Q)^2}e^{-2\int Qdx},$$

unde fit

$$V = \int e^{-2\int Qdx}dx,$$

ita ut integrale sit

$$\int e^{-2\int Qdx}dx - \frac{e^{-2\int Qdx}}{y-Q} = C.$$

COROLLARIUM 3

532. Proposita ergo aequatione $dy + yydx + Xdx = 0$ si eius integrale particulare quoddam constet $y = Q$, ut sit $dQ + QQdx + Xdx = 0$ ideoque

$dy + yydx - dQ - QQdx = 0$, multiplicator pro ea erit $\frac{1}{(y-Q)^2}e^{-2\int Qdx}$ et integrale completum

$$Ce^{2\int Qdx} + \frac{1}{y-Q} = e^{2\int Qdx}\int e^{-2\int Qdx}dx.$$

SCHOLION

533. Aequatio autem in praecedente scholio [§ 529] inventa

$$dy + \frac{yydS}{4QS} + dQ - \frac{(QQ+S)dS}{4QS} = 0$$

non multum habet in recessu; posito enim $y + Q = z$ prodit

$$dz - \frac{zdS}{2S} + \frac{dS(zz-S)}{4QS} = 0;$$

in qua ut bini priores termini in unum contrahantur, ponatur $z = v\sqrt{S}$ reperieturque

$$dv\sqrt{S} + \frac{vvdS}{4Q} - \frac{dS}{4Q} = 0 \quad \text{seu} \quad \frac{dv}{vv-1} + \frac{dS}{4Q\sqrt{S}} = 0;$$

quae cum sit separata, integrale erit

$$\frac{1}{2}l\frac{1+v}{1-v} = \frac{1}{4}\int\frac{dS}{Q\sqrt{S}},$$

ubi est $v = \frac{y+Q}{\sqrt{S}}$.

Aequatio autem in ipsa solutione [§ 528] inventa

$$dy + yydx + \frac{Cdx}{S^4} - \frac{ddS}{Sdx} = 0,$$

ubi S est functio quaecunque ipsius x et $\frac{ddS}{dx} = d.\frac{dS}{dx}$, magis ardua videtur, dum per se fit integrabilis, si dividatur per

$$SSyy - \frac{2SydS}{dx} + \frac{dS^2}{dx^2} + \frac{C}{SS} = \left(Sy - \frac{dS}{dx}\right)^2 + \frac{C}{SS}.$$

At sumto x constante integrale reperitur

$$\frac{1}{\sqrt{C}}\text{Arc. tang.}\frac{SSydx - SdS}{dx\sqrt{C}} + V = \text{Const.};$$

nunc ergo ad functionem V inveniendam sumatur differentiale posita y constante, quod est

$$\frac{2SydS - \frac{SddS}{dx} - \frac{dS^2}{dx}}{SS\left(Sy - \frac{dS}{dx}\right)^2 + C} + dV$$

et aequari debet alteri parti

$$\frac{\frac{Cdx}{S^4} - \frac{ddS}{Sdx} + yydx}{\left(Sy - \frac{dS}{dx}\right)^2 + \frac{C}{SS}} = \frac{\frac{Cdx}{SS} - \frac{SddS}{dx} + SSyydx}{SS\left(Sy - \frac{dS}{dx}\right)^2 + C}.$$

Ergo

$$dV = \frac{SSyydx - 2SydS + \frac{dS^2}{dx} + \frac{Cdx}{SS}}{SS\left(Sy - \frac{dS}{dx}\right)^2 + C} = \frac{dx}{SS}.$$

Quocirca integrale completum est

$$\frac{1}{\sqrt{C}}\text{Arc. tang.}\frac{SSydx - SdS}{dx\sqrt{C}} + \int\frac{dx}{SS} = D.$$

Quodsi sumamus $S = x$, huius aequationis [§ 491]

$$dy + yydx + \frac{Cdx}{x^4} = 0$$

integrale completum est

$$\frac{1}{\sqrt{C}}\text{Arc. tang.}\frac{xxy - x}{\sqrt{C}} - \frac{1}{x} = D.$$

Sin autem sit $S = x^n$, ob $\frac{dS}{dx} = nx^{n-1}$ et $d.\frac{dS}{dx} = n(n-1)x^{n-2}dx$ integrari poterit haec aequatio

$$dy + yydx + \frac{Cdx}{x^{4n}} - \frac{n(n-1)dx}{xx} = 0;$$

integrale enim erit

$$\frac{1}{\sqrt{C}}\text{Arc. tang.}\frac{x^{2n}y - nx^{2n-1}}{\sqrt{C}} - \frac{1}{(2n-1)x^{2n-1}} = D.$$

Supra autem [§ 436] invenimus hanc aequationem

$$dy + yydx + Cx^m dx = 0$$

ad separationem reduci posse, quoties fuerit $m = \frac{-4i}{2i \pm 1}$; iisdem ergo casibus functionem S assignare licebit, ut fiat

$$\frac{C}{S^4} - \frac{ddS}{Sdx^2} = Cx^m;$$

quod cum ad aequationes differentiales secundi gradus pertineat, hic non attingemus.

PROBLEMA 69

534. *Definire functiones P et Q ambarum variabilium x et y, ut aequatio differentialis $Pdx + Qdy = 0$ divisa per $Px + Qy$ fiat per se integrabilis.*

SOLUTIO

Cum formula $\frac{Pdx + Qdy}{Px + Qy}$ debeat esse integrabilis, statuamus $Q = PR$, ut habeamus $\frac{dx + Rdy}{x + Ry}$, sitque $dR = Mdx + Ndy$. Quare fieri oportet

$$\frac{1}{dy} d. \frac{1}{x + Ry} = \frac{1}{dx} d. \frac{R}{x + Ry},$$

unde nanciscimur

$$\frac{-R - Ny}{(x + Ry)^2} = \frac{Mx - R}{(x + Ry)^2}$$

seu $N = -\frac{Mx}{y}$; hinc fit

$$dR = Mdx - \frac{Mxdy}{y} = My \frac{ydx - xdy}{yy};$$

quae formula cum debeat esse integrabilis, necesse est sit My functio ipsius $\frac{x}{y}$, quia $\frac{ydx - xdy}{yy} = d. \frac{x}{y}$, atque ex hac integratione prodit $R = \text{funct.} \frac{x}{y}$ seu, quod eodem redit, R erit functio nullius dimensionis ipsarum x et y. Quocirca cum $\frac{Q}{P} = R$, manifestum est huic conditioni satisfieri, si P et Q fuerint functiones homogeneae eiusdem dimensionum numeri ipsarum x et y; hoc ergo modo eandem integrationem aequationum homogenearum sumus assecuti, quam in capite superiori [§ 477] docuimus.

COROLLARIUM 1

535. Cum igitur $\frac{dt + Rdu}{t + Ru}$ sit integrabile, si fuerit $R = f : \frac{t}{u}$ seu $R = \frac{t}{u} f : \frac{t}{u}$, erit etiam haec formula

$$\frac{\frac{dt}{t} + \frac{du}{u} f : \frac{t}{u}}{1 + f : \frac{t}{u}}$$

integrabilis, quae ita repraesentari potest

$$\frac{\frac{dt}{t} + \frac{du}{u} f : \left(\int \frac{dt}{t} - \int \frac{du}{u}\right)}{1 + f : \left(\int \frac{dt}{t} - \int \frac{du}{u}\right)},$$

ubi littera f denotat functionem quamcunque quantitatis suffixae.

COROLLARIUM 2

536. Ponatur $\frac{dt}{t} = \frac{dx}{X}$ et $\frac{du}{u} = \frac{dy}{Y}$ atque haec formula

$$\frac{\frac{dx}{X} + \frac{dy}{Y} f : \left(\int \frac{dx}{X} - \int \frac{dy}{Y}\right)}{1 + f : \left(\int \frac{dx}{X} - \int \frac{dy}{Y}\right)} = \frac{dx + \frac{X dy}{Y} f : \left(\int \frac{dx}{X} - \int \frac{dy}{Y}\right)}{X + X f : \left(\int \frac{dx}{X} - \int \frac{dy}{Y}\right)}$$

erit per se integrabilis. Quare posito

$$R = \frac{X}{Y} f : \left(\int \frac{dx}{X} - \int \frac{dy}{Y}\right)$$

haec formula $\frac{dx + Rdy}{X + RY}$ erit per se integrabilis, quaecunque functio sit X ipsius x et Y ipsius y.

COROLLARIUM 3

537. Quare si quaerantur functiones P et Q, ut haec aequatio $Pdx + Qdy = 0$ fiat integrabilis, si dividatur per $PX + QY$ existente X functione quacunque ipsius x et Y ipsius y, debet esse

$$\frac{Q}{P} = \frac{X}{Y} \text{funct.} \left(\int \frac{dx}{X} - \int \frac{dy}{Y}\right).$$

COROLLARIUM 4

538. Quare si signa φ et ψ functiones quascunque indicent fueritque

$$P = \frac{V}{X}\varphi : \left(\int\frac{dx}{X} - \int\frac{dy}{Y}\right) \quad \text{et} \quad Q = \frac{V}{Y}\psi : \left(\int\frac{dx}{X} - \int\frac{dy}{Y}\right),$$

haec aequatio $Pdx + Qdy = 0$ integrabilis reddetur, si dividatur per $PX + QY$.

SCHOLION

539. Hinc ergo innumerabiles aequationes proferri possunt, quas integrare licebit, etiamsi alioquin difficillime pateat, quomodo eae ad separationem variabilium reduci queant. Verum haec investigatio proprie ad librum secundum *Calculi Integralis* est referenda, cuius iam egregia specimina hic habentur; definivimus enim functionem R binarum variabilium x et y ex certa conditione inter M et N proposita, scilicet $Mx + Ny = 0$ seu $x\left(\frac{dR}{dx}\right) + y\left(\frac{dR}{dy}\right) = 0$, hoc est ex certa differentialium conditione.[1])

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1) Notandum est magnam partem eorum, quae in Cap. II et III continentur, inveniri in L. EULERI Commentatione 269 (indicis ENESTROEMIANI): *De integratione aequationum differentialium*, Novi comment. acad. sc. Petrop. 8 (1760/1), 1763, p. 3; *LEONHARDI EULERI Opera omnia*, series I, vol. 22. F. E.

CAPUT IV

DE INTEGRATIONE PARTICULARI AEQUATIONUM DIFFERENTIALIUM

DEFINITIO

540. *Integrale particulare aequationis differentialis est relatio variabilium aequationi satisfaciens, quae nullam novam quantitatem constantem in se complectitur. Opponitur ergo integrali completo, quod constantem in differentiali non contentam involvit, in quo tamen contineatur necesse est.*

COROLLARIUM 1

541. Cognito ergo integrali completo ex eo innumerabilia integralia particularia exhiberi possunt, prout constanti illi arbitrariae alii atque alii valores determinati tribuuntur.

COROLLARIUM 2

542. Proposita ergo aequatione differentiali inter variabiles x et y omnes functiones ipsius x, quae loco y substitutae aequationi satisfaciunt, dabunt integralia particularia, nisi forte sint completa.

COROLLARIUM 3

543. Cum omnis aequatio differentialis ad hanc formam $\frac{dy}{dx} = V$ revocetur existente V functione quacunque ipsarum x et y, si eiusmodi constet relatio inter x et y, unde pro $\frac{dy}{dx}$ et V resultent valores aequales, ea pro integrali particulari erit habenda.

SCHOLION 1

544. Interdum facile est integrale particulare quasi divinatione colligere; veluti si proposita sit haec aequatio

$$aady + yydx = aadx + xydx.$$

Statim liquet ei satisfieri ponendo $y = x$; quae relatio cum non solum nullam novam constantem, sed ne eam quidem a, quae in ipsa aequatione differentiali continetur, implicet, utique est integrale particulare; unde nihil pro integrali completo colligere licet.

Saepenumero quidem cognitio integralis particularis ad inventionem completi viam patefacit, quemadmodum in hoc ipso exemplo usu venit; in quo si statuamus $y = x + z$, fit

$$aadx + aadz + xxdx + 2xzdx + zzdx = aadx + xxdx + xzdx$$

seu

$$aadz + xzdx + zzdx = 0,$$

quae aequatio posito $z = \frac{aa}{v}$ abit in hanc $dv - \frac{xvdx}{aa} = dx$, quae per $e^{-\int\frac{xdx}{aa}} = e^{\frac{-xx}{2aa}}$ multiplicata fit integrabilis et dat

$$e^{\frac{-xx}{2aa}}v = \int e^{\frac{-xx}{2aa}}dx \quad \text{seu} \quad v = e^{\frac{xx}{2aa}}\int e^{\frac{-xx}{2aa}}dx,$$

quod ergo est maxime transcendens, cum tamen simplicissimum illud particulare involvat; scilicet si constans integratione $\int e^{\frac{-xx}{2aa}}dx$ invecta sumatur infinita, fit $v = \infty$ et $z = 0$, unde $y = x$.

Interdum autem integrale particulare parum iuvat ad completum investigandum, veluti si habeatur haec aequatio

$$a^3dy + y^3dx = a^3dx + x^3dx,$$

cui manifesto satisfacit $y = x$; posito autem $y = x + z$ prodit

$$a^3dz + 3xxzdx + 3xzzdx + z^3dx = 0,$$

cuius resolutio haud facilior videtur quam illius.

SCHOLION 2

545. In his exemplis integrale particulare statim in oculos incurrit; dantur autem casus, quibus difficilius perspicitur; et quanquam raro inde via pateat ad integrale completum perveniendi, tamen saepenumero plurimum interest integrale particulare nosse, cum eo nonnunquam totum negotium confici possit. Iam enim animadvertimus in omnibus problematibus, quorum solutio ad aequationem differentialem perducitur, constantem arbitrariam per integrationem invectam ex ipsis conditionibus cuique problemati adiunctis determinari, ita ut semper integrali tantum particulari sit opus; quare si eveniat, ut hoc ipsum integrale particulare cognosci possit sine subsidio completi, solutio problematis exhiberi poterit, etiamsi integratio aequationis differentialis non sit in potestate. Quibus ergo casibus sine integratione vera solutio inveniri est censenda, propterea quod proprie loquendo nulla aequatio differentialis integrari existimatur, nisi eius integrale completum assignetur. Quocirca utile erit eos casus perpendere, quibus integrale particulare exhibere licet.

SCHOLION 3

546. Maximi autem est momenti hic animadvertisse non omnes valores aequationi cuipiam differentiali satisfacientes pro eius integrali particulari haberi posse. Veluti si habeatur haec aequatio

$$dy = \frac{dx}{\sqrt{(a-x)}} \quad \text{seu} \quad \frac{dx}{dy} = \sqrt{(a-x)},$$

posito $x = a$ fit tam $\sqrt{(a-x)} = 0$ quam $\frac{dx}{dy} = 0$, ita ut aequatio $x = a$ illi differentiali satisfaciat, cum tamen nequaquam eius sit integrale particulare. Integrale namque completum est

$$y = C - 2\sqrt{(a-x)} \quad \text{seu} \quad a - x = \tfrac{1}{4}(C-y)^2,$$

unde, quicunque valor constanti C tribuatur, nunquam sequitur $a - x = 0$. Simili modo huic aequationi

$$dy = \frac{xdx + ydy}{\sqrt{(xx+yy-aa)}}$$

satisfacit haec aequatio finita $xx+yy=aa$, quae tamen inter integralia particularia admitti nequit, propterea quod in integrali completo

$$y=C+\sqrt{(xx+yy-aa)}$$

neutiquam continetur.

Quare ad integrale particulare non sufficit, ut eo aequationi differentiali satisfiat, sed insuper hanc conditionem adiungi oportet, ut in integrali completo contineatur; ex quo investigatio integralium particularium maxime est lubrica, nisi simul integrale completum innotescat; hoc autem cognito supervacuum esset methodo peculiari in integralia particularia inquirere. Tum enim potissimum iuvat ad investigationem integralium particularium confugere, quando integrale completum elicere non licet. Quo igitur hinc fructum percipere queamus, criteria tradi conveniet, ex quibus valores, qui aequationi cuipiam differentiali satisfaciunt, diiudicare liceat, utrum sint integralia particularia necne. Etiamsi scilicet omnia integralia sint eiusmodi valores, qui aequationi differentiali satisfaciant, tamen non vicissim omnes valores, qui satisfaciunt, sunt integralia. Quod cum parum adhuc sit animadversum, operam dabo, ut hoc argumentum dilucide evolvam.

PROBLEMA 70

547. *Si in aequatione differentiali* $dy=\frac{dx}{Q}$ *functio* Q *evanescat posito* $x=a$, *determinare, quibus casibus haec aequatio* $x=a$ *sit integrale particulare aequationis differentialis propositae.*

SOLUTIO

Cum sit $Q=\frac{dx}{dy}$, posito $x=a$ fit tam $Q=0$ quam $\frac{dx}{dy}=0$, unde hic valor $x=a$ aequationi differentiali propositae $dy=\frac{dx}{Q}$ utique satisfacit; neque tamen hinc sequitur eum esse integrale. Hoc solum scilicet non sufficit, sed insuper requiritur, ut aequatio $x=a$ in integrali completo contineatur, si quidem constanti per integrationem invectae certus quidam valor tribuatur. Ponamus ergo P esse integrale formulae $\frac{dx}{Q}$, ut integrale completum sit $y=C+P$; cui aequationi ponendo $x=a$ satisfieri nequit, nisi posito $x=a$ fiat $P=\infty$; tum enim sumta constante C pariter infinita positione $x=a$ quantitas y manet indeterminata, ideoque si posito $x=a$ fiat $P=\infty$, tum demum aequatio $x = a$ pro integrali particulari erit habenda. En ergo criterium, ex quo dignoscere licet, utrum valor $x = a$ aequationi differentiali $dy = \frac{dx}{Q}$ satisfaciens simul sit eius integrale particulare necne; scilicet tum demum erit integrale, si posito $x = a$ non solum fiat $Q = 0$, sed etiam integrale $P = \int \frac{dx}{Q}$ abeat in infinitum.

Quod quo clarius exponamus, quoniam posito $x = a$ fit $Q = 0$, ponamus $Q = (a - x)^n R$ denotante n numerum quemcunque positivum, et cum aequatio

$$dy = \frac{dx}{Q} = \frac{dx}{(a-x)^n R}$$

induere queat hanc formam

$$dy = \frac{\alpha dx}{(a-x)^n} + \frac{\beta dx}{(a-x)^{n-1}} + \frac{\gamma dx}{(a-x)^{n-2}} + \cdots + \frac{S dx}{R},$$

ratio illius infiniti P pendebit a termino $\int \frac{\alpha dx}{(a-x)^n}$; qui si posito $x = a$ evadat infinitus, etiam integrale $P = \int \frac{dx}{Q}$ erit infinitum, utcunque se habeant reliqua membra. At est $\int \frac{\alpha dx}{(a-x)^n} = \frac{\alpha}{(n-1)(a-x)^{n-1}}$, quae expressio fit infinita posito $x = a$, dummodo $n - 1$ sit numerus positivus vel etiam $n = 1$. Quare dummodo exponens n non sit unitate minor posito $Q = (a - x)^n R$, aequatio $x = a$ pro integrali particulari erit habenda.

COROLLARIUM 1

548. Quoties ergo posito $Q = (a - x)^n R$ exponens n est unitate minor, aequationi $dy = \frac{dx}{Q}$ non convenit integrale particulare $x = a$, etiamsi hoc modo aequationi differentiali satisfiat.

COROLLARIUM 2

549. Si exponens n est unitate minor, formula $\frac{dQ}{dx}$ fit infinita posito $x = a$, unde novum criterium adipiscimur. Scilicet proposita aequatione $dy = \frac{dx}{Q}$ si posito $x = a$ fiat quidem $Q = 0$, at $\frac{dQ}{dx} = \infty$, tum valor $x = a$ non est integrale particulare illius aequationis.

COROLLARIUM 3

550. His igitur casibus exclusis aequationis $dy = \frac{dx}{Q}$, ubi posito $x = a$ fit $Q = 0$, integrale particulare semper erit $x = a$, nisi eodem casu $x = a$ fiat $\frac{dQ}{dx} = \infty$; hoc est, quoties valor formulae $\frac{dQ}{dx}$ fuerit vel finitus vel evanescat.

SCHOLION 1

551. Haec conclusio inversioni propositionum hypotheticarum innixa licet videri queat suspecta ac regulis Logicae adversa, verum totum ratiocinium regulis apprime est consentaneum, cum a sublatione consequentis ad sublationem antecedentis concludat. Quoties enim posito $Q = (a - x)^n R$ exponens n est unitate minor, toties $\frac{dQ}{dx}$ fit $= \infty$ posito $x = a$. Quare si posito $x = a$ non fiat $\frac{dQ}{dx} = \infty$ ideoque eius valor vel finitus vel evanescat, tum certe exponens n non est unitate minor; erit ergo vel maior unitate vel ipsi aequalis, utroque autem casu integrale $P = \int \frac{dx}{Q}$ posito $x = a$ fit infinitum ideoque aequatio $x = a$ est integrale particulare. Quare si in aequatione differentiali $dy = \frac{dx}{Q}$ posito $x = a$ fiat $Q = 0$, examinatur valor $\frac{dQ}{dx}$ pro casu $x = a$; qui si fuerit vel finitus vel evanescat, aequatio $x = a$ est integrale particulare; sin autem is sit infinitus, ea inter integralia locum non habet, etiamsi aequationi differentiali satisfiat. Eadem regula quoque locum habet, si aequatio differentialis fuerit huiusmodi $dy = \frac{Pdx}{Q}$ seu $\frac{dy}{dx} = \frac{P}{Q}$ ac posito $x = a$ fiat $Q = 0$, quaecunque fuerit P functio ipsarum x et y; quin etiam necesse non est, ut Q sit functio solius variabilis x, sed simul alteram y utcunque implicare potest.

SCHOLION 2

552. Demonstratio quidem inde est petita, quod quantitas Q, quae posito $x = a$ evanescit, factorem implicet potestatem quampiam ipsius $a - x$, quod in functionibus algebraicis est manifestum. Verum in functionibus transcendentibus eadem regula locum habet, cum potestate talibus dignitatibus aequivaleant. Veluti si sit

$$dy = \frac{dx}{lx - la},$$

ubi $Q = lx - la = l\frac{x}{a}$ fitque $Q = 0$ posito $x = a$, quaeratur $\frac{dQ}{dx} = \frac{1}{x}$; quae formula cum non fiat infinita posito $x = a$, integrale particulare erit $x = a$.

Quod etiam valet pro aequatione

$$dy = \frac{Pdx}{lx - la},$$

dummodo P non fiat $= 0$ posito $x = a$. Sit enim $P = \frac{1}{x}$; erit integrando $y = C + l(lx - la)$ et $l\frac{x}{a} = e^{y-C}$. Sumta iam constante $C = \infty$ fit $l\frac{x}{a} = 0$ ideoque $x = a$, quod ergo est integrale particulare. Simili modo si sit

$$dy = Pdx : (e^{\frac{x}{a}} - e),$$

ubi $Q = e^{\frac{x}{a}} - e$ ideoque posito $x = a$ fit $Q = 0$, quia $\frac{dQ}{dx} = \frac{1}{a}e^{\frac{x}{a}}$ hincque posito $x = a$ fit $\frac{dQ}{dx} = \frac{e}{a}$, erit $x = a$ etiam integrale particulare. Sumatur $P = e^{\frac{x}{a}}$, ut integratio succedat, et quia $y = C + al(e^{\frac{x}{a}} - e)$ hincque $e^{\frac{x}{a}} = e + e^{\frac{y-C}{a}}$, statuatur $C = \infty$; erit $e^{\frac{x}{a}} = e$ ideoque $x = a$, quod ergo manifesto est integrale particulare.

EXEMPLUM 1

553. *Proposita aequatione differentiali* $dy = \frac{Pdx}{\sqrt{S}}$, *in qua* S *evanescat posito* $x = a$, *definire casus, quibus aequatio* $x = a$ *est eius integrale particulare.*

Cum hic sit $\sqrt{S} = Q$, erit $dQ = \frac{dS}{2\sqrt{S}}$; ergo ut integrale particulare sit $x = a$, necesse est, ut posito $x = a$ fiat $\frac{dQ}{dx} = \frac{dS}{2dx\sqrt{S}}$ quantitas finita. Hinc eodem casu quantitas $\frac{dS^2}{Sdx^2}$ fieri debet finita, unde, cum S evanescat, etiam $\frac{dS^2}{dx^2}$ ac proinde $\frac{dS}{dx}$ evanescere debet. Tum autem posito $x = a$ illius fractionis valor est

$$\frac{2dSddS}{dSdx^2} = \frac{2ddS}{dx^2},$$

quem ergo finitum esse oportet vel $= 0$.

Quare ut aequatio $x = a$ sit integrale particulare aequationis propositae, hae conditiones requiruntur, primo ut posito $x = a$ fiat $S = 0$, secundo ut fiat $\frac{dS}{dx} = 0$, ac tertio ut huius formulae $\frac{ddS}{dx^2}$ valor prodeat vel finitus vel $= 0$, dummodo ne fiat infinite magnus. Si S sit functio rationalis, haec eo redeunt, ut S factorem habeat $(a - x)^2$ vel potestatem altiorem.

SCHOLION

554. Haec resolutio usum habet in motu corporis ad centrum virium attracti dignoscendo, num in circulo fiat. Si enim distantia corporis a centro ponatur $= x$ et vis centripeta huic distantiae conveniens $= X$, pro tempore t talis reperitur aequatio[1])

$$dt = \frac{x\,dx}{\sqrt{(Exx - c^4 - 2\alpha xx\int X dx)}},$$

ubi E est constans per praecedentem integrationem ingressa, cuius valor quaeritur, ut hinc aequationi satisfaciat valor $x = a$, quo casu corpus in circulo revolvetur. Hic ergo est $S = Exx - c^4 - 2\alpha xx\int X dx$ vel sumi potest

$$S = E - \frac{c^4}{xx} - 2\alpha\int X dx.$$

Non solum ergo haec quantitas, sed etiam eius differentiale

$$\frac{dS}{dx} = \frac{2c^4}{x^3} - 2\alpha X$$

evanescere debet posito $x = a$, neque tamen differentio-differentiale

$$\frac{ddS}{dx^2} = -\frac{6c^4}{x^4} - \frac{2\alpha dX}{dx}$$

in infinitum abire debet. Inde ergo constans a erit valor ipsius x ex hac aequatione $\alpha x^3 X = c^4$ resultans, qui est radius circuli, in quo corpus revolvi poterit, dummodo constans E, a qua celeritas pendet, ita fuerit comparata, ut posito $x = a$ fiat $E = \frac{c^4}{aa} + 2\alpha\int X dx$; nisi forte eodem casu expressio $\frac{6c^4}{x^4} + \frac{2\alpha dX}{dx}$ seu saltem haec $\frac{dX}{dx}$ fiat infinita. Hoc enim si eveniret, motus in circulo tolleretur; ad quod ostendendum ponamus

$$X = b + \sqrt{(a - x)},$$

ut

$$\frac{dX}{dx} = -\frac{1}{2\sqrt{(a-x)}}$$

1) L. EULERI *Mechanica*, t. I § 601; LEONHARDI EULERI *Opera omnia*, series II, vol. 1, p. 199. F. E.

fiat infinitum posito $x=a$, et aequatio $\alpha x^3 X = c^4$ dabit $\alpha a^3 b = c^4$. Tum vero ob

$$\int X dx = bx - \tfrac{2}{3}(a-x)^{\frac{3}{2}}$$

erit $E = \alpha ab + 2\alpha ab = 3\alpha ab$ nostraque aequatio fit

$$dt = \frac{x dx}{\sqrt{\left(3\alpha abxx - \alpha a^3 b - 2\alpha bx^3 + \frac{4}{3}\alpha xx(a-x)^{\frac{3}{2}}\right)}},$$

cui valor $x=a$ certe non convenit tanquam integrale. Fit enim

$$S = \alpha(a-x)(-aab - abx + 2bxx + \tfrac{4}{3}xx\sqrt{(a-x)}),$$

cuius factor cum non sit $(a-x)^2$, sed tantum $(a-x)^{\frac{3}{2}}$, integrale particulare $x=a$ locum habere nequit.

EXEMPLUM 2

555. *Proposita aequatione differentiali* $dy = \frac{P dx}{\sqrt[n]{S^m}}$, *in qua* S *evanescat posito* $x=a$, *invenire casus, quibus integrale particulare est* $x=a$.

Cum fiat $S=0$ posito $x=a$, concipere licet $S=(a-x)^\lambda R$ eritque denominator

$$\sqrt[n]{S^m} = (a-x)^{\frac{\lambda m}{n}} R^{\frac{m}{n}},$$

unde patet aequationem $x=a$ fore integrale particulare aequationis propositae, si fuerit $\frac{\lambda m}{n}$ numerus positivus unitate maior seu saltem unitati aequalis, hoc est, si sit vel $\lambda = \frac{n}{m}$ vel $\lambda > \frac{n}{m}$, quae diiudicatio, si S sit functio algebraica, facillime instituitur. Sin autem sit transcendens, ut exponens λ in numeris exhiberi nequeat, uti licebit altera regula; scilicet cum sit $\sqrt[n]{S^m} = Q$, erit $\frac{dQ}{dx} = \frac{mS^{\frac{m-n}{n}} dS}{ndx}$, cuius valor debet esse finitus vel nullus posito $x=a$, siquidem integrale sit $x=a$. Sit igitur quoque necesse est hoc casu quantitas $\frac{S^{m-n} dS^n}{dx^n}$ finita. Quaeratur ergo huius formulae valor casu $x=a$, qui si prodeat infinite magnus, aequatio $x=a$ non erit integrale, sin autem sit vel finitus vel nullus, erit ea certe integrale particulare aequationis propositae. Hic duo constituendi sunt casus, prout fuerit vel $m>n$ vel $m<n$.

I. Si $m > n$, quia posito $x = a$ fit $S^{m-n} = 0$, nisi eodem casu fiat $\frac{dS}{dx} = \infty$, certe erit $x = a$ integrale. Sin autem fiat $\frac{dS}{dx} = \infty$, utrumque evenire potest, ut sit integrale et ut non sit. Ad quod dignoscendum ponatur $\frac{dx}{dS} = T$, ut nostra formula evadat $S^{m-n} : T^n$, cuius tam numerator quam denominator evanescit posito $x = a$, ex quo eius valor reducitur ad

$$\frac{(m-n)S^{m-n-1}dS}{nT^{n-1}dT} = \frac{-(m-n)S^{m-n-1}dS^{n+2}}{ndx^n ddS},$$

qui si sit vel finitus vel nullus, integrale erit $x = a$. Simili modo ulterius progredi licet distinguendo casus $m > n + 1$ et $m < n + 1$.

II. Si $m < n$, formula nostra erit $\frac{dS^n}{S^{n-m}dx^n}$, cuius valor ut fiat finitus, necesse est, ut sit $\frac{dS}{dx} = 0$, ac praeterea, quia numerator ac denominator posito $x = a$ evanescit, formulae nostrae valor erit

$$= \frac{ndS^{n-1}ddS}{(n-m)S^{n-m-1}dSdx^n} = \frac{ndS^{n-2}ddS}{(n-m)S^{n-m-1}dx^n},$$

quem finitum esse oportet.

Facillime autem iudicium absolvetur ponendo statim $x = a + \omega$; cum enim posito $x = a$ fiat $S = 0$, hac substitutione quantitas S semper resolvi poterit in huiusmodi formam $P\omega^\alpha + Q\omega^\beta + R\omega^\gamma +$ etc., cuius tantum unus terminus $P\omega^\alpha$ infimam potestatem ipsius ω complectens spectetur; ac si fuerit vel $\alpha = \frac{n}{m}$ vel $\alpha > \frac{n}{m}$, aequatio $x = a$ certe erit integrale particulare.

SCHOLION

556. Haec ultima methodus est tutissima ac semper etiam in formulis transcendentibus optimo successu adhiberi potest. Scilicet proposita aequatione $dy = \frac{Pdx}{Q}$, in qua posito $x = a$ fiat $Q = 0$ neque vero etiam numerator P evanescat, statuatur $x = a \pm \omega$ et quantitas ω spectetur ut infinite parva, ut omnes eius potestates prae infima evanescant, atque quantitas Q huiusmodi formam $R\omega^\lambda$ accipiet, ex qua patebit, nisi exponens λ unitate fuerit minor, aequationem $x = a$ certe fore integrale particulare aequationis propositae. Veluti si habeamus

$$dy = \frac{dx}{\sqrt{\left(1 + \cos.\frac{\pi x}{a}\right)}},$$

cuius denominator evanescit sumto $x=a$ ob $\cos.\pi = -1$, ponamus $x=a-\omega$; erit $\cos.\frac{\pi x}{a} = \cos.\left(\pi - \frac{\pi\omega}{a}\right) = -1 + \frac{\pi\pi\omega\omega}{2aa}$ ob ω infinite parvum, hinc nostrae aequationis denominator fiet $=\frac{\pi\omega}{a\sqrt{2}}$, unde concludimus integrale particulare utique esse $x=a$. Non autem foret integrale huius aequationis

$$dy = \frac{dx}{\sqrt[3]{\left(1+\cos.\frac{\pi x}{a}\right)}}.$$

PROBLEMA 71

557. *Proposita aequatione differentiali, in qua variabiles sunt a se invicem separatae, investigare eius integralia particularia.*

SOLUTIO

Sit proposita haec aequatio $\frac{dx}{X} = \frac{dy}{Y}$, in qua X sit functio ipsius x et Y ipsius y tantum. Ac primo ponatur $X=0$ indeque quaerantur valores ipsius x, quorum quisque sit $x=a$, ita ut posito $x=a$ fiat $X=0$; tum examinetur valor formulae $\frac{dX}{dx}$ posito $x=a$, qui nisi fiat infinitus, aequationis propositae integrale particulare certe erit $x=a$. Vel ponatur $x=a\pm\omega$ spectando ω ut quantitatem infinite parvam, ac si prodeat $X=P\omega^\lambda$, exponens λ, nisi sit unitate minor, indicabit integrale $x=a$; sin autem sit unitate minor, aequatio $x=a$ pro integrali non erit habenda.

Simili modo examinetur alterius partis denominator Y; qui si evanescat posito $y=b$ hocque casu formula $\frac{dY}{dy}$ non fiat infinita, aequatio $y=b$ erit integrale particulare; quod ergo etiam evenit, si posito $y=b\pm\omega$ prodeat $Y=Q\omega^\lambda$, ubi exponens λ unitate non sit minor.

COROLLARIUM 1

558. Nisi ergo membra aequationis separatae fuerint fractiones, quarum denominatores certis casibus evanescant, huiusmodi integralia particularia non dantur, nisi forte in tali aequatione $Pdx = Qdy$ factores P et Q certis casibus fiant infiniti, qui autem casus ad praecedentem facile reducitur.

COROLLARIUM 2

559. Veluti si habeatur

$$dx \operatorname{tang.} \frac{\pi x}{2a} = \frac{dy}{b-y},$$

primo quidem integrale particulare est $y = b$, tum vero, quia posito $x = a$ fit $\operatorname{tang.} \frac{\pi x}{2a} = \infty$, prius membrum ita exhibeatur $dx : \operatorname{cot.} \frac{\pi x}{2a}$, cuius denominator posito $x = a - \omega$ fit $\operatorname{cot.} \left(\frac{\pi}{2} - \frac{\pi\omega}{2a}\right) = \operatorname{tang.} \frac{\pi\omega}{2a} = \frac{\pi\omega}{2a}$, ubi cum exponens ipsius ω unitate non sit minor, aequatio $x = a$ erit quoque integrale particulare.

COROLLARIUM 3

560. Hinc ergo interdum pro eadem aequatione duo plurave integralia particularia assignari possunt. Veluti pro hac aequatione

$$\frac{m\,dx}{a-x} = \frac{n\,dy}{b-y}$$

integralia particularia sunt $a - x = 0$ et $b - y = 0$, quae etiam ex integrali completo $(a-x)^m = C(b-y)^n$ consequuntur, illud sumendo $C = 0$, hoc vero sumendo $C = \infty$.

COROLLARIUM 4

561. Simili modo huius aequationis

$$\frac{ma\,dx}{aa-xx} = \frac{nb\,dy}{bb-yy}$$

quatuor dantur integralia particularia $a + x = 0$, $a - x = 0$, $b + y = 0$, $b - y = 0$. Integrale completum vero est

$$\frac{m}{2}\, l\frac{a+x}{a-x} = \frac{1}{2}\, C + \frac{n}{2}\, l\frac{b+y}{b-y} \quad \text{seu} \quad \left(\frac{a+x}{a-x}\right)^m = C\left(\frac{b+y}{b-y}\right)^n$$

vel

$$(a+x)^m(b-y)^n = C(a-x)^m(b+y)^n,$$

unde illa sponte fluunt.

COROLLARIUM 5

562. Hinc patet, si fuerit

$$dy = \frac{Pdx}{(a+x)^\alpha (b+x)^\beta (c+x)^\gamma},$$

integralia particularia fore $a+x=0$, $b+x=0$, $c+x=0$, si modo exponentes α, β, γ non fuerint unitate minores. Quare si Q sit functio rationalis ipsius x, proposita aequatione $dy = \frac{Pdx}{Q}$ omnes factores ipsius Q nihilo aequales positi praebent integralia particularia.

SCHOLION 1

563. Hoc etiam pro factoribus imaginariis valet, etiamsi inde parum lucri nanciscamur. Si enim proposita sit aequatio

$$dy = \frac{adx}{aa+xx},$$

ex denominatore $aa+xx$ oriuntur integralia particularia

$$x = a\sqrt{-1} \quad \text{et} \quad x = -a\sqrt{-1},$$

quae ex integrali completo, quod est $y = C + \text{Ang. tang. } \frac{x}{a}$, minus sequi videntur. Verum posito $x = a\sqrt{-1}$ notandum est esse Ang. tang. $\sqrt{-1} = \infty\sqrt{-1}$, unde si constanti C similis forma signo contrario affecta tribuatur, altera quantitas y manet indeterminata, etiamsi ponatur $x = a\sqrt{-1}$, quae positio propterea pro integrali particulari est habenda. Est enim in genere

$$\text{Ang. tang. } u\sqrt{-1} = \int \frac{du\sqrt{-1}}{1-uu} = \frac{\sqrt{-1}}{2} l \frac{1+u}{1-u},$$

unde posito $u = +1$ vel $u = -1$ prodit $\infty\sqrt{-1}$, quod infinitum in causa est, ut integralia assignata locum habeant.

Quocirca in genere affirmare licet, si fuerit $dy = \frac{Pdx}{Q}$ denominatorque Q factorem habeat $(a+x)^\lambda$, cuius exponens λ unitate non sit minor, semper aequationem $a+x=0$ fore integrale particulare. Sin autem λ sit unitate minor, etsi positivus, non erit $a+x=0$ integrale particulare, etiamsi posito $x=-a$ aequationi differentiali satisfaciat.

SCHOLION 2

564. Insigne hoc est paradoxon a nemine adhuc[1]), quantum mihi quidem constat, observatum, quod aequationi differentiali eiusmodi valor satisfacere queat, qui tamen eius non sit integrale; atque adeo vix patet, quomodo haec cum solita integralium idea conciliari possint. Quoties enim proposita aequatione differentiali eiusmodi relationem variabilium exhibere licet, quae ibi substituta satisfaciat seu aequationem identicam producat, vix cuiquam in mentem venit dubitare, an illa relatio pro integrali saltem particulari sit habenda, cum tamen hinc proclive sit in errorem delabi. Veluti etiamsi huic aequationi

$$dy\sqrt{(aa-xx-yy)}=xdx+ydy$$

satisfaciat haec aequatio finita $xx+yy=aa$, tamen enormem errorem committeremus, si eam pro integrali particulari habere vellemus, propterea quod ea in integrali completo $y=C-\sqrt{(aa-xx-yy)}$ neutiquam continetur. Quamobrem etsi omne integrale aequationi differentiali satisfacere debet, tamen non vicissim concludere licet omnem aequationem finitam, quae satisfaciat, eius esse integrale; verum praeterea requiritur, ut ea certa quadam proprietate sit praedita, cuiusmodi hic exposuimus et qua demum efficitur, ut in integrali completo contineatur. Hoc autem minime adversatur verae integralium notioni, quam hic stabilivimus, neque huiusmodi dubium unquam in integralia per certas regulas inventa cadere potest, sed tantum in eiusmodi integralibus, quae divinando quasi sumus assecuti, locum habet. Saepenumero autem, quando integratio non succedit, divinationi plurimum tribui solet; tum igitur maxime cavendum est, ne relationem quampiam satisfacientem temere pro integrali particulari proferamus. Quod cum iam in aequationibus separatis simus assecuti, quomodo in omnibus aequationibus differentialibus huiusmodi errores vitari oporteat, sedulo investigemus.

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1) Revera iam BROOK TAYLOR (1685—1731) hoc observavit in libro *Methodus incrementorum directa et inversa*, Londini 1715, p. 26, et A. C. CLAIRAUT (1713—1765) aequationes huius generis consideravit in Commentatione *Solution de plusieurs problèmes, où il s'agit de trouver des courbes dont la propriété consiste dans une certaine relation entre leurs branches, exprimée par une équation donnée*, Mém. de l'acad. d. sc. de Paris 1734 (1736), p. 196, imprimis p. 209. F. E.

PROBLEMA 72

565. *Si quaepiam relatio inter binas variabiles satisfaciat aequationi differentiali, definire, utrum ea sit integrale particulare necne.*

SOLUTIO

Sit $Pdx = Qdy$ aequatio differentialis proposita, ubi P et Q sint functiones quaecunque ipsarum x et y, cui satisfaciat relatio quaepiam inter x et y, ex qua fiat $y = X$, functioni scilicet cuidam ipsius x, ita ut, si loco y ubique scribatur X, revera prodeat $Pdx = Qdy$ seu $\frac{dy}{dx} = \frac{P}{Q}$. Quaeritur ergo, utrum hic valor $y = X$ pro integrali aequationis propositae haberi possit necne.

Ad hoc diiudicandum ponatur $y = X + \omega$ fietque $\frac{dX}{dx} + \frac{d\omega}{dx} = \frac{P}{Q}$, ubi notetur, si esset $\omega = 0$, fore $\frac{dX}{dx} = \frac{P}{Q}$. Quare ob ω expressio $\frac{P}{Q}$ hac substitutione reducetur ad $\frac{dX}{dx}$ una cum quantitate ita per ω affecta, ut evanescat posito $\omega = 0$. In hoc negotio sufficit ω ut particulam infinite parvam spectasse, cuius ergo potestates altiores prae infima negligere liceat. Ponamus igitur hinc fieri

$$\frac{P}{Q} = \frac{dX}{dx} + S\omega^{\lambda}$$

habebiturque $\frac{d\omega}{dx} = S\omega^{\lambda}$ seu $\frac{d\omega}{\omega^{\lambda}} = Sdx$. Ex superioribus iam perspicuum est tum demum fore $y = X$ integrale particulare seu $\omega = 0$, cum exponens λ fuerit unitati aequalis vel maior; similis enim hic est ratio ac supra, qua requiritur, ut integrale $\int Sdx = \int \frac{d\omega}{\omega^{\lambda}}$ fiat infinitum casu proposito, quo $\omega = 0$; hoc autem non evenit, nisi λ sit unitati aequalis vel > 1.

Quodsi ergo aequationi $Pdx = Qdy$ seu $\frac{dy}{dx} = \frac{P}{Q}$ satisfaciat valor $y = X$, statuatur $y = X + \omega$ spectata particula ω infinite parva et investigetur hinc forma

$$\frac{P}{Q} = \frac{dX}{dx} + S\omega^{\lambda},$$

ex qua, nisi sit $\lambda < 1$, concludetur illum valorem $y = X$ esse integrale particulare aequationis propositae.

SCHOLION

566. Cum ω tractetur ut quantitas infinite parva, valor ipsius $\frac{P}{Q}$ posito $y = X + \omega$ per differentiationem commodissime inveniri posse videtur. Cum enim $\frac{P}{Q}$ sit functio ipsarum x et y, statuamus $d.\frac{P}{Q} = Mdx + Ndy$, et quia posito $y = X$ fractio $\frac{P}{Q}$ abit in $\frac{dX}{dx}$ per hypothesin, si loco y scribatur $X + \omega$, ea in $\frac{dX}{dx} + N\omega$ transibit, unde ob exponentem ipsius ω unitatem sequeretur aequationem $y = X$ semper esse integrale particulare, quod tamen secus evenire potest. Ex quo patet differentiationem loco substitutionis adhiberi non posse; quod quo clarius ostendatur, ponamus esse $\frac{P}{Q} = \sqrt{(y - X)} + \frac{dX}{dx}$, unde posito $y = X + \omega$ manifesto oritur $\frac{P}{Q} = \frac{dX}{dx} + \sqrt{\omega}$. At differentiatione utentes ponendo $d.\frac{P}{Q} = Mdx + Ndy$ fiet $N = \frac{1}{2\sqrt{(y - X)}}$ hincque $\frac{P}{Q} = \frac{dX}{dx} + N\omega$, quae expressio ab illa discrepat. Illa scilicet aequationem $y = X$ ex integralium numero removet, haec vero admittere videtur. Verum et hic notandum est quantitatem N ipsam potestatem ipsius ω negative involvere, unde potestas ω deprimatur. Quare ne hanc rationem spectare opus sit, semper praestat vera substitutione uti differentiatione seposita. Hoc observato haud difficile erit omnes valores, qui aequationi cuipiam differentiali satisfaciunt, diiudicare, utrum sint vera integralia necne.

EXEMPLUM 1

567. *Cum huic aequationi* $dx(1 - y^m)^n = dy(1 - x^m)^n$ *manifesto satisfaciat* $y = x$, *utrum sit eius integrale particulare necne, definire.*

Ponatur $y = x + \omega$ et spectato ω ut quantitate minima est

$$y^m = x^m + mx^{m-1}\omega$$

et

$$(1 - y^m)^n = (1 - x^m - mx^{m-1}\omega)^n = (1 - x^m)^n - mnx^{m-1}\omega(1 - x^m)^{n-1},$$

unde aequatio $\frac{dy}{dx} = \frac{(1 - y^m)^n}{(1 - x^m)^n}$ abit in

$$1 + \frac{d\omega}{dx} = 1 - \frac{mnx^{m-1}\omega}{1 - x^m} \quad \text{seu} \quad \frac{d\omega}{\omega} = -\frac{mnx^{m-1}dx}{1 - x^m},$$

ubi cum ω habeat dimensionem integram, aequatio $y = x$ certe est integrale particulare aequationis differentialis propositae.

EXEMPLUM 2

568. *Cum huic aequationi* $ady - adx = dx\sqrt{(yy - xx)}$ *satisfaciat valor* $y = x$, *investigare, utrum is sit eius integrale particulare necne.*

Ponatur $y = x + \omega$ et sumta ω quantitate infinite parva, cum sit $\sqrt{(yy - xx)} = \sqrt{2x\omega}$, erit $ad\omega = dx\sqrt{2x\omega}$ seu $\frac{ad\omega}{\sqrt{\omega}} = dx\sqrt{2x}$. Quoniam igitur hic $d\omega$ dividitur per potestatem ipsius ω, cuius exponens est unitate minor, sequitur valorem $y = x$ non esse integrale particulare aequationis propositae, etiamsi ei satisfaciat. Scilicet si eius integrale completum exhibere liceret, pateret, quomodocunque constans arbitraria per integrationem ingressa definiretur, in ea aequationem $y = x$ non contentum iri.

SCHOLION

569. Hinc nova ratio intelligitur, cur diiudicatio integralis ab exponente ipsius ω pendeat. Cum enim in exemplo proposito facto $y = x + \omega$ prodeat $\frac{ad\omega}{\sqrt{\omega}} = dx\sqrt{2x}$, erit integrando $2a\sqrt{\omega} = C + \frac{2}{3}x\sqrt{2x}$. Verum per hypothesin ω est quantitas infinite parva, hinc autem, utcunque definiatur constans C, quantitas ω obtinet valorem finitum, qui adeo quantumvis magnus evadere potest; quod cum hypothesi adversetur, necessario sequitur aequationem $y = x$ integrale esse non posse hocque semper evenire debere, quoties $d\omega$ prodit divisum per potestatem ipsius ω, cuius exponens unitate est minor. Contra vero patet, si facta substitutione exposita prodeat $\frac{d\omega}{\omega} = Rdx$, ut posito $\int Rdx = S$ fiat $l\omega = lC + lS$ seu $\omega = CS$, sumta constante C evanescente utique ipsam quantitatem ω evanescere; quod idem evenit, si prodeat $\frac{d\omega}{\omega^\lambda} = Rdx$ existente $\lambda > 1$. Erit enim $\frac{1}{(\lambda - 1)\omega^{\lambda-1}} = C - S$ seu $(\lambda - 1)\omega^{\lambda-1} = \frac{1}{C - S}$, unde sumto $C = \infty$ quantitas ω revera fit evanescens, ut hypothesis exigit.

Caeterum aequatio huius exempli posito $x = pp - qq$ et $y = pp + qq$ ab irrationalitate liberatur fitque

$$4aqdq = 4pq(pdp - qdq) \quad \text{sive} \quad adq = ppdp - pqdq,$$

quae nullo modo tractari posse videtur; neque ergo eius integrale completum exhiberi potest. Cui aequationi cum non amplius satisfacit $x = y$ seu $q = 0$, hinc quoque concludendum est valorem $y = x$ non esse integrale particulare.

EXEMPLUM 3

570. *Cum huic aequationi* $aady - aadx = dx(yy - xx)$ *satisfaciat valor* $y = x$, *investigare, utrum is sit eius integrale particulare necne.*

Ponatur $y = x + \omega$ spectata ω ut quantitate infinite parva et ob $yy - xx = 2x\omega$ aequatio nostra hanc induet formam $aad\omega = 2x\omega dx$ seu $\frac{aad\omega}{\omega} = 2xdx$. Quia igitur hic $d\omega$ dividitur per potestatem primam ipsius ω, aequatio $y = x$ utique erit integrale particulare aequationis propositae atque adeo etiam in integrali completo continetur. Hoc enim invenitur ponendo $y = x - \frac{aa}{u}$, quo fit

$$\frac{a^4 du}{uu} = dx\left(\frac{a^4}{uu} - \frac{2aax}{u}\right) \quad \text{seu} \quad du + \frac{2uxdx}{aa} = dx.$$

Multiplicetur per $e^{\frac{xx}{aa}}$ et integrale prodit

$$e^{\frac{xx}{aa}}u = C + \int e^{\frac{xx}{aa}}dx \quad \text{hincque} \quad y = x - aae^{\frac{xx}{aa}}:\left(C + \int e^{\frac{xx}{aa}}dx\right).$$

Quodsi ergo constans C capiatur infinita, fit $y = x$.

SCHOLION

571. Si in hac aequatione ut supra ponatur $x = pp - qq$ et $y = pp + qq$, oritur $aadq = 2ppq(pdp - qdq)$, cui satisfacit $q = 0$, unde casus $y = x$ nascitur. At facta hac transformatione difficulter patet, quomodo eius integrale inveniri oporteat. Si quidem superiorem reductionem perpendamus, intelligemus hanc aequationem integrabilem reddi, si multiplicetur per $e^{(pp-qq)^2:aa}:q^3$; quod cum per se haud facile pateat, consultum erit hac substitutione uti $pp - qq = rr$, qua fit $pp = qq + rr$ et $pdp - qdq = rdr$, unde aequatio abit in

$$aadq = 2qrdr(qq + rr) \quad \text{seu} \quad \frac{aadq}{2q^3} = rdr + \frac{r^3dr}{qq},$$

quae posito $\frac{1}{qq} = s$ facile integratur.

Quoties ergo licet eiusmodi relationem inter variabiles colligere, quae aequationi differentiali satisfaciat, hoc modo iudicari poterit, utrum ea relatio pro integrali particulari sit habenda necne. Pro inventione autem huiusmodi integralium particularium regulae vix tradi possunt; quae enim habentur regulae aeque ad integralia completa invenienda patent. Ita quae supra circa aequationes separatas observavimus, ob id ipsum, quod sunt separatae, via simul ad integrale completum est patefacta. Simili modo si altera methodus per factores succedat, plerumque ex ipsis factoribus, quibus aequatio integrabilis redditur, integralia particularia concludi possunt; quemadmodum in sequentibus propositionibus declarabimus.

THEOREMA

572. *Si aequatio differentialis* $Pdx + Qdy = 0$ *per functionem* M *multiplicata reddatur integrabilis, integrale particulare erit* $M = 0$, *nisi eodem casu* P *vel* Q *abeat in infinitum.*

DEMONSTRATIO

Ponamus u esse factorem ipsius M et ostendendum est aequationem $u = 0$ esse integrale particulare aequationis propositae. Cum u aequetur certae functioni ipsarum x et y, definiatur inde altera variabilis y, ut aequatio prodeat inter binas variabiles x et u, quae sit $Rdx + Sdu = 0$, unde posito multiplicatore $M = Nu$ integrabilis erit haec forma

$$NRudx + NSudu = 0.$$

Quodsi iam neque R neque S per u dividatur, quo casu posito $u = 0$ neque P neque Q abit in infinitum, integrale utique per u erit divisibile. Nam sive id colligatur ex termino $NRudx$ spectata u ut constante sive ex termino $NSudu$ spectata x constante, integrale prodit factorem u implicans, si quidem in integratione constans omittatur. Unde concludimus integrale completum huiusmodi formam esse habiturum $Vu = C$. Quare si haec constans C nihilo aequalis capiatur, integrale particulare erit $u = 0$, iis scilicet casibus exceptis, quibus functiones R et S iam ipsae per u essent divisae ideoque ratiocinium nostrum vim suam amitteret. His ergo casibus exclusis, quoties aequatio $Pdx + Qdy = 0$ per functionem M multiplicata fit per se integrabilis eaque functio M factorem habeat u, integrale particulare erit $u = 0$, quod similiter de singulis factoribus functionis M valet.

SCHOLION

573. Limitatio adiecta absolute est necessaria, cum ea neglecta universum ratiocinium claudicet. Quod quo facilius intelligatur, consideremus hanc aequationem

$$\frac{adx}{y-x}+dy-dx=0,$$

quae per $y-x$ multiplicata manifesto fit integrabilis; ponamus ergo hunc multiplicatorem $y-x=u$ seu $y=x+u$, unde nostra aequatio erit $\frac{adx}{u}+du=0$, quae per u multiplicata abit in $adx+udu=0$; ubi cum pars adx non per u sit multiplicata, neutiquam concludere licet integrale per u fore divisibile, quippe quod est $ax+\frac{1}{2}uu$. Hinc patet, si modo pars dx per u esset multiplicata, etiamsi altera pars du factore u careret, tamen integrale per u divisibile fore, veluti evenit in $udx+xdu$, cuius integrale xu utique factorem habet u. Ex quo intelligitur, si formula $Pudx+Qdu$ fuerit per se integrabilis, dummodo Q non dividatur per u vel per potestatem eius prima altiorem, etiam integrale omissa scilicet constante fore per u divisibile.

THEOREMA

574. *Si aequatio differentialis $Pdx+Qdy=0$ per functionem M divisa evadat per se integrabilis, integrale particulare erit $M=0$, nisi posito $M=0$ vel P vel Q evanescat.*

DEMONSTRATIO

Habeat divisor M factorem u, ut sit $M=Nu$, et ostendi oportet integrale particulare futurum $u=0$, id quod de singulis factoribus divisoris M, siquidem plures habeat, est tenendum. Cum igitur u sit functio ipsarum x et y, definiatur inde altera y per x et u, ut prodeat huiusmodi aequatio $Rdx+Sdu=0$, quae ergo per Nu divisa per se erit integrabilis. Quaeri igitur oportet integrale formulae $\frac{Rdx}{Nu}+\frac{Sdu}{Nu}$, ubi assumimus neque R neque S per u multiplicari neque hoc modo factorem u ex denominatore tolli. Quodsi iam hoc integrale ex solo membro $\frac{Rdx}{Nu}$ colligatur spectando u ut constantem, prodit id $\frac{1}{u}\int\frac{Rdx}{N}+f:u$; sin autem ex altero membro $\frac{Sdu}{Nu}$ sumta x constante colligatur, quia S non factorem habet u, id semper ita erit comparatum, ut posito $u=0$ fiat infinitum. Ex quo integrale, quod sit V, ita erit comparatum, ut fiat $=\infty$ posito $u=0$; quare cum integrale completum futurum sit $V=C$, huic aequationi sumta constante C infinita satisfit ponendo $u=0$. Concludimus itaque, si divisor $M=Nu$ reddat aequationem differentialem $Pdx+Qdy=0$ per se integrabilem, ex quolibet divisoris M factore u obtineri integrale particulare $u=0$, nisi forte posito $u=0$ quantitates P et Q vel R et S evanescant.

COROLLARIUM 1

575. Si aequatio $Pdx+Qdy=0$ fuerit homogenea, ea, ut supra vidimus, integrabilis redditur, si dividatur per $Px+Qy$, quare integrale eius particulare erit $Px+Qy=0$. Quae aequatio cum etiam sit homogenea, factores habebit formae $\alpha x+\beta y$, quorum quisque nihilo aequatus dabit integrale particulare.

COROLLARIUM 2

576. Pro hac aequatione

$$ydx(c+nx)-dy(y+a+bx+nxx)=0$$

divisorem, quo integrabilis redditur, supra (§ 488) exhibuimus, unde integrale particulare concluditur $y=0$, tum vero

$$nyy+(2na-bc)y+n(b-2c)xy+(na+cc-bc)(a+bx+nxx)=0,$$

cuius radices sunt

$$ny=\frac{1}{2}bc-na+n\left(c-\frac{1}{2}b\right)x\pm(c+nx)\sqrt{\left(\frac{1}{4}bb-na\right)}.$$

COROLLARIUM 3

577. Pro hac aequatione differentiali

$$\frac{ndx(1+yy)\sqrt{(1+yy)}}{\sqrt{(1+xx)}}+(x-y)dy=0$$

divisorem, quo integrabilis redditur, supra (§ 489) dedimus, unde integrale particulare concludimus $x - y + n\sqrt{(1 + xx)(1 + yy)} = 0$ seu

$$yy - 2xy + xx = nn + nnxx + nnyy + nnxxyy,$$

ex quo porro fit

$$y = \frac{x \pm n(1 + xx)\sqrt{(1 - nn)}}{1 - nn(1 + xx)}.$$

COROLLARIUM 4

578. Pro hac aequatione differentiali

$$dy + yydx - \frac{adx}{x^4} = 0$$

multiplicatorem supra (§ 491) invenimus $\frac{xx}{xx(1 - xy)^2 - a}$, unde integrale particulare concludimus $xx(1 - xy)^2 - a = 0$ hincque

$$x(1 - xy) = \pm\sqrt{a} \quad \text{seu} \quad y = \frac{1}{x} \pm \frac{\sqrt{a}}{xx},$$

ita ut bina habeamus integralia particularia, quae autem imaginaria evadunt, si a fuerit quantitas negativa.

SCHOLION

579. Haec fere sunt, quae circa tractationem aequationum differentialium adhuc sunt explorata, nonnulla tamen subsidia evolutio aequationum differentialium secundi gradus infra suppeditabit. Huc autem commode referri possunt, quae circa comparationem certarum formularum transcendentium haud ita pridem sunt investigata. Quemadmodum enim logarithmi et arcus circulares, etsi sunt quantitates transcendentes, inter se comparari atque adeo aeque ac quantitates algebraicae in calculo tractari possunt, ita similem comparationem inter certas quantitates transcendentes altioris generis instituere licet, quae scilicet continentur in formula hac

$$\int \frac{dx}{\sqrt{(A + Bx + Cx^2 + Dx^3 + Ex^4)}},$$

ubi etiam numerator rationalis veluti $\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 +$ etc. addi potest. Quod argumentum cum sit maxime arduum atque adeo vires Analyseos superare videatur, nisi certa ratione expediatur, in Analysin inde haud spernenda incrementa redundant; imprimis autem resolutio aequationum differentialium non mediocriter perfici videtur. Cum enim proposita fuerit huiusmodi aequatio

$$\frac{dx}{\sqrt{(A+Bx+Cx^2+Dx^3+Ex^4)}}=\frac{dy}{\sqrt{(A+By+Cy^2+Dy^3+Ey^4)}},$$

statim quidem patet eius integrale particulare $x=y$, verum integrale completum maxime transcendens fore videtur, cum utraque formula per se neque ad logarithmos neque ad arcus circulares reduci queat. Quare eo magis erit mirandum, quod integrale completum per aequationem adeo algebraicam inter x et y exhiberi possit. Quo autem methodus ad haec sublimia ducens clarius perspiciatur, eam primo ad quantitates transcendentes notas hac formula

$$\int\frac{dx}{\sqrt{(A+Bx+Cxx)}}$$

contentas applicemus, deinceps eius usum in formulis illis magis complexis ostensuri.

CAPUT V

DE COMPARATIONE QUANTITATUM TRANSCENDENTIUM IN FORMA $\int\frac{Pdx}{\sqrt{(A+2Bx+Cxx)}}$ CONTENTARUM[1])

PROBLEMA 73

580. *Proposita inter x et y hac aequatione algebraica*

$$\alpha+2\beta(x+y)+\gamma(xx+yy)+2\delta xy=0$$

invenire formulas integrales formae praescriptae, quae inter se comparari queant.

SOLUTIO

Differentietur aequatio proposita et ex eius differentiali

$$2\beta dx+2\beta dy+2\gamma xdx+2\gamma ydy+2\delta xdy+2\delta ydx=0$$

colligetur haec aequatio

$$dx(\beta+\gamma x+\delta y)+dy(\beta+\gamma y+\delta x)=0.$$

Statuatur

$$\beta+\gamma x+\delta y=p \quad \text{et} \quad \beta+\gamma y+\delta x=q$$

1) Vide L. EULERI Commentationem 263 (indicis ENESTROEMIANI): *Specimen novae methodi curvarum quadraturas et rectificationes aliasque quantitates transcendentes inter se comparandi,* Novi comment. acad. sc. Petrop. 7 (1758/9), 1761, p. 83; *LEONHARDI EULERI Opera omnia,* series I, vol. 20, p. 108. Vide porro Commentationem 818 (indicis ENESTROEMIANI): *De comparatione arcuum curvarum irrectificabilium,* Opera postuma 1, Petropoli 1862, p. 452; *LEONHARDI EULERI Opera omnia,* series I, vol. 21, p. 296. F. E.

atque ex priori erit

$$pp = \beta\beta + 2\beta\gamma x + 2\beta\delta y + \gamma\gamma xx + 2\gamma\delta xy + \delta\delta yy,$$

a qua subtrahatur aequatio proposita per γ multiplicata

$$0 = \alpha\gamma + 2\beta\gamma x + 2\gamma\beta y + \gamma\gamma xx + \gamma\gamma yy + 2\gamma\delta xy$$

fietque

$$pp = \beta\beta - \alpha\gamma + 2\beta(\delta - \gamma)y + (\delta\delta - \gamma\gamma)yy;$$

similique modo reperietur

$$qq = \beta\beta - \alpha\gamma + 2\beta(\delta - \gamma)x + (\delta\delta - \gamma\gamma)xx,$$

unde erit $pdx + qdy = 0$. Cum iam sit p functio ipsius y et q similis functio ipsius x, ponatur

$$\beta\beta - \alpha\gamma = A, \quad \beta(\delta - \gamma) = B \quad \text{et} \quad \delta\delta - \gamma\gamma = C,$$

unde colligitur

$$\delta - \gamma = \frac{B}{\beta} \quad \text{et} \quad \delta + \gamma = \frac{C}{\delta - \gamma} = \frac{\beta C}{B}$$

hincque

$$\delta = \frac{BB + \beta\beta C}{2B\beta} \quad \text{et} \quad \gamma = \frac{\beta\beta C - BB}{2B\beta};$$

prima vero dat

$$\alpha = \frac{\beta\beta - A}{\gamma} = \frac{2B\beta(\beta\beta - A)}{\beta\beta C - BB}.$$

Quibus valoribus pro α, γ, δ assumtis aequatio $\frac{dx}{q} + \frac{dy}{p} = 0$ abit in hanc

$$\frac{dx}{\sqrt{(A + 2Bx + Cxx)}} + \frac{dy}{\sqrt{(A + 2By + Cyy)}} = 0,$$

cui ergo aequationi differentiali satisfacit aequatio

$$\frac{2B\beta(\beta\beta - A)}{\beta\beta C - BB} + 2\beta(x + y) + \frac{\beta\beta C - BB}{2B\beta}(xx + yy) + \frac{BB + \beta\beta C}{B\beta}xy = 0;$$

quae cum contineat constantem novam β, erit adeo integrale completum aequationis differentialis inventae.

Neque vero opus est, ut formulae illae ipsis litteris A, B, C aequentur, sed sufficit, ut ipsis sint proportionales, unde fit

$$\frac{\beta\beta-\alpha\gamma}{\beta(\delta-\gamma)}=\frac{A}{B} \quad \text{et} \quad \frac{\delta+\gamma}{\beta}=\frac{C}{B}.$$

Ergo

$$\delta=\frac{\beta C}{B}-\gamma \quad \text{et} \quad \alpha=\frac{\beta\beta}{\gamma}-\frac{\beta A}{\gamma B}(\delta-\gamma) \quad \text{seu} \quad \alpha=\frac{\beta\beta}{\gamma}-\frac{\beta\beta AC}{\gamma BB}+\frac{2\beta A}{B}.$$

Quare aequationis differentialis

$$\frac{dx}{\sqrt{(A+2Bx+Cxx)}}+\frac{dy}{\sqrt{(A+2By+Cyy)}}=0$$

integrale completum est

$$\beta\beta(BB-AC)+2\beta\gamma AB+2\beta\gamma BB(x+y)+\gamma\gamma BB(xx+yy)$$
$$+2\gamma B(\beta C-\gamma B)xy=0,$$

ubi ratio $\frac{\beta}{\gamma}$ constantem arbitrariam exhibet.

COROLLARIUM 1

581. Ex aequatione proposita radicem extrahendo fit

$$y=\frac{-\beta-\delta x+\sqrt{(\beta\beta+2\beta\delta x+\delta\delta xx-\alpha\gamma-2\beta\gamma x-\gamma\gamma xx)}}{\gamma}$$

seu loco α et δ substitutis valoribus

$$y=-\frac{\beta}{\gamma}-\frac{\beta C-\gamma B}{\gamma B}x+\sqrt{\frac{\beta\beta C-2\beta\gamma B}{\gamma\gamma BB}}(A+2Bx+Cxx).$$

COROLLARIUM 2

582. Si ergo $x=0$, fit

$$y=-\frac{\beta}{\gamma}+\sqrt{\frac{\beta\beta AC-2\beta\gamma AB}{\gamma\gamma BB}};$$

ponatur hic valor $=a$, ut sit

$$\gamma Ba+\beta B=\sqrt{(\beta\beta AC-2\beta\gamma AB)},$$

unde sumtis quadratis oritur

$$\gamma\gamma BBaa + 2\beta\gamma BBa + \beta\beta BB = \beta\beta AC - 2\beta\gamma AB$$

hincque

$$\frac{\gamma}{\beta} = \frac{-A - Ba + \sqrt{A(A + 2Ba + Caa)}}{Baa}$$

seu

$$\frac{\beta}{\gamma} = \frac{B(A + Ba + \sqrt{A(A + 2Ba + Caa)})}{AC - BB}.$$

SCHOLION 1

583. Ut aequatio assumta

$$\alpha + 2\beta(x + y) + \gamma(xx + yy) + 2\delta xy = 0$$

satisfaciat aequationi differentiali

$$\frac{dx}{\sqrt{(A + 2Bx + Cxx)}} + \frac{dy}{\sqrt{(A + 2By + Cyy)}} = 0,$$

necesse est, ut sit

$$\beta\beta - \alpha\gamma = mA, \quad \beta(\delta - \gamma) = mB \quad \text{et} \quad \delta\delta - \gamma\gamma = mC,$$

unde fit

$$\beta + \gamma y + \delta x = \sqrt{m(A + 2Bx + Cxx)}$$

et

$$\beta + \gamma x + \delta y = \sqrt{m(A + 2By + Cyy)}.$$

At ex datis A, B, C litterarum α, β, γ, δ et m tres tantum definiuntur; quare cum binae maneant indeterminatae, aequatio assumta, etiamsi per quemvis coefficientium dividatur, unam tamen constantem continet novam, ex quo ea pro integrali completo erit habenda. Quare etsi aequationis differentialis neutra pars integrationem algebraice admittit, tamen integrale completum algebraice exhiberi potest. Loco constantis arbitrariae is valor ipsius y introduci potest, quem recipit posito $x = 0$; cum autem evenire possit, ut hic valor fiat imaginarius, conveniet istam constantem ita definiri, ut posito $x = a$ fiat $y = b$, quo pacto ad omnes casus applicatio fieri poterit. Hinc erit

$$\frac{\beta + \gamma b + \delta a}{\beta + \gamma a + \delta b} = \sqrt{\frac{A + 2Ba + Caa}{A + 2Bb + Cbb}},$$

unde colligitur

$$\beta = \frac{(\gamma a + \delta b)\sqrt{(A+2Ba+Caa)} - (\gamma b + \delta a)\sqrt{(A+2Bb+Cbb)}}{-\sqrt{(A+2Ba+Caa)} + \sqrt{(A+2Bb+Cbb)}}$$

et

$$\sqrt{m}(A+2Ba+Caa) = \frac{(\delta-\gamma)(b-a)\sqrt{(A+2Ba+Caa)}}{\sqrt{(A+2Bb+Cbb)} - \sqrt{(A+2Ba+Caa)}}$$

seu

$$\sqrt{m} = \frac{(\delta-\gamma)(b-a)}{\sqrt{(A+2Bb+Cbb)} - \sqrt{(A+2Ba+Caa)}}.$$

Ponatur brevitatis gratia

$$\sqrt{(A+2Ba+Caa)} = \mathfrak{A} \quad \text{et} \quad \sqrt{(A+2Bb+Cbb)} = \mathfrak{B},$$

ut sit

$$\sqrt{m} = \frac{(\delta-\gamma)(b-a)}{\mathfrak{B}-\mathfrak{A}} \quad \text{et} \quad \beta = \frac{\mathfrak{A}(\gamma a + \delta b) - \mathfrak{B}(\gamma b + \delta a)}{\mathfrak{B}-\mathfrak{A}},$$

et aequatio $\beta(\delta-\gamma) = mB$ induet hanc formam

$$\mathfrak{A}(\gamma a + \delta b) - \mathfrak{B}(\gamma b + \delta a) = \frac{B(\delta-\gamma)(b-a)^2}{\mathfrak{B}-\mathfrak{A}},$$

unde fit

$$\left.\begin{aligned} &+\gamma\mathfrak{A}\mathfrak{B} - \gamma A - \gamma B(a+b) - \gamma C(aa-ab+bb) \\ &+\delta\mathfrak{A}\mathfrak{B} - \delta A - \delta B(a+b) - \delta Cab \end{aligned}\right\} = 0.$$

Statuatur ergo

$$\gamma = n\mathfrak{A}\mathfrak{B} - nA - nB(a+b) - nCab,$$

$$\delta = nA + nB(a+b) + nC(aa-ab+bb) - n\mathfrak{A}\mathfrak{B},$$

$$\sqrt{m} = \frac{n(b-a)(\mathfrak{A}^2+\mathfrak{B}^2-2\mathfrak{A}\mathfrak{B})}{\mathfrak{B}-\mathfrak{A}} = n(b-a)(\mathfrak{B}-\mathfrak{A}),$$

$$\beta = nB(b-a)^2, \quad \text{ergo} \quad \delta-\gamma = \frac{m}{n(b-a)^2},$$

unde cum sit $\delta+\gamma = nC(b-a)^2$, erit utique $\delta\delta-\gamma\gamma = mC$. Superest, ut fiat $\alpha\gamma = \beta\beta - mA$, hoc est

$$\alpha\gamma = nnBB(b-a)^4 - nnA(b-a)^2(\mathfrak{B}-\mathfrak{A})^2$$

seu

$$\alpha\gamma = nn(b-a)^2(BB(b-a)^2 - A(\mathfrak{B}-\mathfrak{A})^2).$$

Vel cum posito $x=a$ fiat $y=b$, erit quoque

$$\alpha = -2\beta(a+b) - \gamma(aa+bb) - 2\delta ab$$

hincque

$$\alpha = n(a-b)^2(A - B(a+b) - Cab - \mathfrak{A}\mathfrak{B}),$$

unde aequatio nostra assumta est

$$\begin{gathered}(b-a)^2(A - B(a+b) - Cab - \mathfrak{A}\mathfrak{B})\\ +2B(b-a)^2(x+y) - (A + B(a+b) + Cab - \mathfrak{A}\mathfrak{B})(xx+yy)\\ +2(A + B(a+b) + C(aa-ab+bb) - \mathfrak{A}\mathfrak{B})xy = 0.\end{gathered}$$

SCHOLION 2

584. Si ponatur $\beta=0$, ut aequatio sit

$$\alpha + \gamma(xx+yy) + 2\delta xy = 0,$$

erit

$$y = \frac{-\delta x + \sqrt{(-\alpha\gamma + (\delta\delta - \gamma\gamma)xx)}}{\gamma}.$$

Posito ergo $-\alpha\gamma = mA$ et $\delta\delta - \gamma\gamma = mC$, ut sit $\gamma y + \delta x = \sqrt{m(A+Cxx)}$, erit

$$\frac{dx}{\sqrt{(A+Cxx)}} + \frac{dy}{\sqrt{(A+Cyy)}} = 0,$$

cuius aequationis integrale completum erit ipsa aequatio assumta, pro qua habebitur $\frac{C}{A} = \frac{\gamma\gamma - \delta\delta}{\alpha\gamma}$ seu $\delta = \sqrt{\left(\gamma\gamma - \frac{\alpha\gamma C}{A}\right)}$. Sin autem posito $x=0$ fieri debeat $y=b$, ob $\gamma b = \sqrt{mA}$ erit $\gamma = \frac{\sqrt{mA}}{b}$; tum $\alpha = -b\sqrt{mA}$ et $\delta = \sqrt{\left(\frac{mA}{bb} + mC\right)}$. Habebitur ergo haec aequatio

$$\frac{y\sqrt{mA}}{b} + \frac{x\sqrt{m(A+Cbb)}}{b} = \sqrt{m(A+Cxx)},$$

quae praebet

$$y = -x\sqrt{\frac{A+Cbb}{A}} + b\sqrt{\frac{A+Cxx}{A}},$$

quae est integrale completum aequationis illius differentialis. Quare si x capiatur negative, huius aequationis differentialis

$$\frac{dx}{\sqrt{(A+Cxx)}} = \frac{dy}{\sqrt{(A+Cyy)}}$$

integrale completum est

$$y = x\sqrt{\frac{A+Cbb}{A}} + b\sqrt{\frac{A+Cxx}{A}}.$$

Quodsi simili modo calculus in genere tractetur, aequationis differentialis

$$\frac{dx}{\sqrt{(A+2Bx+Cxx)}} + \frac{dy}{\sqrt{(A+2By+Cyy)}} = 0,$$

si brevitatis gratia ponatur $\sqrt{(A+2Bb+Cbb)} = \mathfrak{B}$, erit integrale completum

$$y\left(\sqrt{A} + \frac{Bb}{\sqrt{A}-\mathfrak{B}}\right) + x\left(\mathfrak{B} + \frac{Bb}{\sqrt{A}-\mathfrak{B}}\right) = \frac{Bbb}{\sqrt{A}-\mathfrak{B}} + b\sqrt{(A+2Bx+Cxx)},$$

unde casus praecedens manifesto sequitur, si ponatur $B=0$. Verum ope levis substitutionis hae formulae, ubi adest B, ad illum casum, ubi $B=0$, reduci possunt.

PROBLEMA 74

585. *Si $\Pi : z$ significet eam functionem ipsius z, quae oritur ex integratione formulae $\int \frac{dz}{\sqrt{(A+Czz)}}$ integrali hoc ita sumto, ut evanescat posito $z=0$, comparationem inter huiusmodi functiones instituere.*

SOLUTIO

Consideretur haec aequatio differentialis

$$\frac{dx}{\sqrt{(A+Cxx)}} = \frac{dy}{\sqrt{(A+Cyy)}},$$

unde, cum sit per hypothesin

$$\int \frac{dx}{\sqrt{(A+Cxx)}} = \Pi : x \quad \text{et} \quad \int \frac{dy}{\sqrt{(A+Cyy)}} = \Pi : y$$

utroque integrali ita sumto, ut evanescat illud posito $x=0$, hoc vero posito $y=0$, integrale completum erit

$$\Pi : y = \Pi : x + C.$$

Ante autem vidimus hoc integrale esse

$$y = x\sqrt{\frac{A + Cbb}{A}} + b\sqrt{\frac{A + Cxx}{A}},$$

ubi posito $x = 0$ fit $y = b$; quare, cum $\Pi : 0 = 0$, erit

$$\Pi : y = \Pi : x + \Pi : b,$$

cui ergo aequationi transcendentali satisfacit haec algebraica

$$y = x\sqrt{\frac{A + Cbb}{A}} + b\sqrt{\frac{A + Cxx}{A}}.$$

Simili modo sumto b negativo haec aequatio

$$\Pi : y = \Pi : x - \Pi : b.$$

convenit cum hac

$$y = x\sqrt{\frac{A + Cbb}{A}} - b\sqrt{\frac{A + Cxx}{A}}$$

sicque tam summa quam differentia duarum huiusmodi functionum per similem functionem exprimi potest. Hic iam nullo habito discrimine inter quantitates variabiles et constantes, dum $\Pi : z$ functionem determinatam ipsius z significat, scilicet

$$\Pi : z = \int \frac{dz}{\sqrt{(A + Czz)}},$$

quae, ut assumsimus, evanescat posito $z = 0$, ut hoc signandi modo recepto sit

$$\Pi : r = \Pi : p + \Pi : q,$$

debet esse

$$r = p\sqrt{\frac{A + Cqq}{A}} + q\sqrt{\frac{A + Cpp}{A}};$$

ut vero sit

$$\Pi : r = \Pi : p - \Pi : q,$$

debet esse

$$r = p\sqrt{\frac{A + Cqq}{A}} - q\sqrt{\frac{A + Cpp}{A}},$$

utrinque autem sublata irrationalitate prodit inter p, q, r haec aequatio

$$p^4 + q^4 + r^4 - 2ppqq - 2pprr - 2qqrr = \frac{4Cppqqrr}{A},$$

cuius forma hanc suppeditat proprietatem, ut, si p, q, r sint latera cuiusdam trianguli eique circumscribatur circulus, cuius diameter vocetur $= T$, semper sit $A + 4CTT = 0$. Illa autem aequatio ob plures quas complectitur radices satisfacit huic relationi

$$\Pi : p \pm \Pi : q \pm \Pi : r = 0.$$

COROLLARIUM 1

586. Hinc statim deducitur nota arcuum circularium comparatio ponendo $A = 1$ et $C = -1$. Tum enim fit

$$\Pi : z = \int \frac{dz}{\sqrt{(1-zz)}} = \text{Ang. sin. } z,$$

hincque ut sit

$$\text{Ang. sin. } r = \text{Ang. sin. } p + \text{Ang. sin. } q,$$

oportet esse

$$r = p\sqrt{(1-qq)} + q\sqrt{(1-pp)},$$

et ut sit

$$\text{Ang. sin. } r = \text{Ang. sin. } p - \text{Ang. sin. } q,$$

debet esse

$$r = p\sqrt{(1-qq)} - q\sqrt{(1-pp)},$$

uti constat.

COROLLARIUM 2

587. Si sit $A = 1$ et $C = 1$, erit

$$\Pi : z = \int \frac{dz}{\sqrt{(1+zz)}} = l\left(z + \sqrt{(1+zz)}\right);$$

unde ut sit

$$l\left(r + \sqrt{(1+rr)}\right) = l\left(p + \sqrt{(1+pp)}\right) + l\left(q + \sqrt{(1+qq)}\right),$$

erit

$$r = p\sqrt{(1+qq)} + q\sqrt{(1+pp)};$$

ut autem sit

$$l\left(r + \sqrt{(1+rr)}\right) = l\left(p + \sqrt{(1+pp)}\right) - l\left(q + \sqrt{(1+qq)}\right),$$

erit

$$r = p\sqrt{(1+qq)} - q\sqrt{(1+pp)},$$

uti ex indole logarithmorum sponte liquet.

COROLLARIUM 3

588. Si ponamus in priori formula generali $q = p$, ut sit

$$\Pi : r = 2\Pi : p,$$

erit

$$r = 2p\sqrt{\frac{A+Cpp}{A}}.$$

Hinc porro, si fiat $q = 2p\sqrt{\frac{A+Cpp}{A}}$, erit $\Pi : r = \Pi : p + 2\Pi : p = 3\Pi : p$ sumto

$$r = p\sqrt{\frac{A+Cqq}{A}} + q\sqrt{\frac{A+Cpp}{A}}.$$

Est vero

$$\sqrt{\frac{A+Cqq}{A}} = \sqrt{\left(1+\frac{4Cpp}{A}\left(1+\frac{Cpp}{A}\right)\right)} = 1 + \frac{2Cpp}{A},$$

unde, ut sit

$$\Pi : r = 3\Pi : p,$$

fit

$$r = p\left(1+\frac{2Cpp}{A}\right) + 2p\left(1+\frac{Cpp}{A}\right) = 3p + \frac{4Cp^3}{A}.$$

SCHOLION

589. Quo haec multiplicatio facilius continuari queat, praeter relationem aequationi

$$\Pi : r = \Pi : p + \Pi : q$$

respondentem, quae est

$$r = p\sqrt{\frac{A+Cqq}{A}} + q\sqrt{\frac{A+Cpp}{A}},$$

notetur aequatio

$$\Pi : p = \Pi : r - \Pi : q,$$

cui respondet relatio

$$p = r\sqrt{\frac{A+Cqq}{A}} - q\sqrt{\frac{A+Crr}{A}},$$

unde fit

$$\sqrt{\frac{A+Crr}{A}}=\frac{r}{q}\sqrt{\frac{A+Cqq}{A}}-\frac{p}{q}=\frac{p}{q}\cdot\frac{A+Cqq}{A}+\sqrt{\left(\frac{A+Cpp}{A}\right)\left(\frac{A+Cqq}{A}\right)}-\frac{p}{q}$$

seu

$$\sqrt{\frac{A+Crr}{A}}=\frac{Cpq}{A}+\sqrt{\left(\frac{A+Cpp}{A}\right)\left(\frac{A+Cqq}{A}\right)}.$$

Quare ut sit

$$\Pi:r=\Pi:p+\Pi:q,$$

habemus non solum

$$r=p\sqrt{\left(1+\frac{C}{A}qq\right)}+q\sqrt{\left(1+\frac{C}{A}pp\right)},$$

sed etiam

$$\sqrt{\left(1+\frac{C}{A}rr\right)}=\frac{C}{A}pq+\sqrt{\left(1+\frac{C}{A}pp\right)\left(1+\frac{C}{A}qq\right)}.$$

Ponamus brevitatis gratia $\sqrt{\left(1+\frac{C}{A}pp\right)}=P$ et sumto $q=p$, ut sit

$$\Pi:r=2\Pi:p,$$

erit

$$r=2Pp \quad \text{et} \quad \sqrt{\left(1+\frac{C}{A}rr\right)}=\frac{C}{A}pp+PP,$$

qui valor ipsius r pro q sumtus dabit

$$\Pi:r=3\Pi:p$$

existente

$$r=\frac{C}{A}p^3+3PPp \quad \text{et} \quad \sqrt{\left(1+\frac{C}{A}rr\right)}=\frac{3C}{A}Ppp+P^3.$$

Hic valor ipsius r denuo pro q sumtus dabit

$$\Pi:r=4\Pi:p$$

existente

$$r=\frac{4C}{A}Pp^3+4P^3p \quad \text{et} \quad \sqrt{\left(1+\frac{C}{A}rr\right)}=\frac{CC}{AA}p^4+\frac{6C}{A}PPpp+P^4.$$

Loco q substituatur hic valor ipsius r, ut prodeat

$$\Pi:r=5\Pi:p$$

existente

$$r=\frac{CC}{AA}p^5+\frac{10C}{A}PPp^3+5P^4p \quad \text{et} \quad \sqrt{\left(1+\frac{C}{A}rr\right)}=\frac{5CC}{AA}Pp^4+\frac{10C}{A}P^3pp+P^5.$$

Atque hinc generatim concludere licet, ut sit

$$\Pi : r = n\Pi : p,$$

esse debere

$$r\sqrt{\frac{C}{A}} = \frac{1}{2}\left(P + p\sqrt{\frac{C}{A}}\right)^n - \frac{1}{2}\left(P - p\sqrt{\frac{C}{A}}\right)^n$$

et

$$\sqrt{\left(1 + \frac{C}{A}rr\right)} = \frac{1}{2}\left(P + p\sqrt{\frac{C}{A}}\right)^n + \frac{1}{2}\left(P - p\sqrt{\frac{C}{A}}\right)^n$$

seu

$$r = \frac{\sqrt{A}}{2\sqrt{C}}\left(P + p\sqrt{\frac{C}{A}}\right)^n - \frac{\sqrt{A}}{2\sqrt{C}}\left(P - p\sqrt{\frac{C}{A}}\right)^n.$$

Haec igitur relatio inter p et r satisfaciet huic aequationi differentiali

$$\frac{dr}{\sqrt{(A + Crr)}} = \frac{ndp}{\sqrt{(A + Cpp)}},$$

dum meminerimus esse $P = \sqrt{\left(1 + \frac{Cpp}{A}\right)}$.

PROBLEMA 75

590. *Si ponatur* $\int \frac{dz}{\sqrt{(A + Czz)}} = \Pi : z$ *integrali ita sumto, ut evanescat posito* $z = f$, *unde* $\Pi : z$ *fit functio determinata ipsius* z, *comparationem inter huiusmodi functiones instituere.*

SOLUTIO

Consideretur haec aequatio differentialis

$$\frac{dx}{\sqrt{(A + Cxx)}} + \frac{dy}{\sqrt{(A + Cyy)}} = 0,$$

unde integrando fit

$$\Pi : x + \Pi : y = \text{Const.}$$

Integrale autem sit quoque [§ 584]

$$\alpha + \gamma(xx + yy) + 2\delta xy = 0,$$

quod ut locum habeat, necesse est sit

$$-\alpha\gamma = Am \quad \text{et} \quad \delta\delta - \gamma\gamma = Cm;$$

tum vero erit

$$\gamma x + \delta y = \sqrt{m(A + Cyy)} \quad \text{et} \quad \gamma y + \delta x = \sqrt{m(A + Cxx)}.$$

Ponamus constantem integratione ingressam ita definiri, ut posito $x = a$ fiat $y = b$, et integrale erit

$$\Pi : x + \Pi : y = \Pi : a + \Pi : b.$$

Pro forma autem algebraica invenienda sit brevitatis gratia

$$\sqrt{(A + Caa)} = \mathfrak{A} \quad \text{et} \quad \sqrt{(A + Cbb)} = \mathfrak{B}$$

eritque

$$\gamma a + \delta b = \mathfrak{B}\sqrt{m} \quad \text{et} \quad \gamma b + \delta a = \mathfrak{A}\sqrt{m},$$

unde colligitur

$$\gamma = \frac{\mathfrak{A}b - \mathfrak{B}a}{bb - aa}\sqrt{m} \quad \text{et} \quad \delta = \frac{\mathfrak{B}b - \mathfrak{A}a}{bb - aa}\sqrt{m}.$$

Quocirca aequatio integralis algebraica erit

$$(\mathfrak{A}b - \mathfrak{B}a)x + (\mathfrak{B}b - \mathfrak{A}a)y = (bb - aa)\sqrt{(A + Cyy)}$$

seu

$$(\mathfrak{A}b - \mathfrak{B}a)y + (\mathfrak{B}b - \mathfrak{A}a)x = (bb - aa)\sqrt{(A + Cxx)}.$$

Hinc y per x ita definitur, ut sit

$$y = \frac{(\mathfrak{A}a - \mathfrak{B}b)x + (bb - aa)\sqrt{(A + Cxx)}}{\mathfrak{A}b - \mathfrak{B}a},$$

quae fractio supra et infra per $\mathfrak{A}b + \mathfrak{B}a$ multiplicando ob

$$\mathfrak{A}\mathfrak{A}bb - \mathfrak{B}\mathfrak{B}aa = A(bb - aa)$$

et

$$(\mathfrak{A}a - \mathfrak{B}b)(\mathfrak{A}b + \mathfrak{B}a) = (\mathfrak{A}\mathfrak{A} - \mathfrak{B}\mathfrak{B})ab - \mathfrak{A}\mathfrak{B}(bb - aa) = -(bb - aa)(Cab + \mathfrak{A}\mathfrak{B})$$

abit in

$$y = -\frac{(Cab + \mathfrak{A}\mathfrak{B})x}{A} + \frac{(\mathfrak{A}b + \mathfrak{B}a)\sqrt{(A + Cxx)}}{A}.$$

Hinc porro colligitur

$$(bb - aa)\sqrt{(A + Cyy)} = (\mathfrak{A}b - \mathfrak{B}a)x - \frac{(\mathfrak{B}b - \mathfrak{A}a)^2}{\mathfrak{A}b - \mathfrak{B}a}x + \frac{(\mathfrak{B}b - \mathfrak{A}a)(bb - aa)}{\mathfrak{A}b - \mathfrak{B}a}\sqrt{(A + Cxx)}$$

seu

$$\sqrt{(A + Cyy)} = -\frac{C(bb - aa)}{\mathfrak{A}b - \mathfrak{B}a}x + \frac{\mathfrak{B}b - \mathfrak{A}a}{\mathfrak{A}b - \mathfrak{B}a}\sqrt{(A + Cxx)},$$

ubi iterum supra et infra multiplicando per $\mathfrak{A}b + \mathfrak{B}a$ fit

$$\sqrt{(A + Cyy)} = -\frac{C(\mathfrak{A}b + \mathfrak{B}a)}{A}x + \frac{(Cab + \mathfrak{A}\mathfrak{B})}{A}\sqrt{(A + Cxx)}.$$

Necesse autem est valorem formulae $\sqrt{(A + Cyy)}$ hoc modo potius definiri quam extractione radicis, qua ambiguitas implicaretur.

Quocirca haec aequatio transcendens

$$\Pi : r + \Pi : s = \Pi : p + \Pi : q$$

praebet sequentem determinationem algebraicam, si quidem brevitatis gratia ponamus

$$\sqrt{(A + Cpp)} = P, \quad \sqrt{(A + Cqq)} = Q \quad \text{et} \quad \sqrt{(A + Crr)} = R;$$

scilicet ut sit $\Pi : s = \Pi : p + \Pi : q - \Pi : r$, erit

$$s = \frac{-PQr - Cpqr + PRq + QRp}{A}$$

et

$$\sqrt{(A + Css)} = \frac{-CPqr - CQpr + CRpq + PQR}{A}$$

seu

$$\sqrt{(A + Css)} = \frac{PQR + C(Rpq - Pqr - Qpr)}{A}.$$

COROLLARIUM 1

591. Quoniam est per hypothesin $\Pi : f = 0$, si ponamus brevitatis gratia $\sqrt{(A + Cff)} = F$ et $r = f$, ut sit $R = F$, haec aequatio

$$\Pi : s = \Pi : p + \Pi : q$$

praebet

$$s = \frac{F(Pq + Qp) - PQf - Cfpq}{A}$$

et

$$\sqrt{(A + Css)} = \frac{FPQ + CFpq - Cf(Pq + Qp)}{A}.$$

COROLLARIUM 2

592. Si ponamus $q = f$ et $Q = F$, ut sit $\Pi : q = 0$, haec aequatio

$$\Pi : s = \Pi : p - \Pi : r$$

praebet

$$s = \frac{F(Rp - Pr) + fPR - Cfpr}{A}$$

et

$$\sqrt{(A + Css)} = \frac{FPR - CFpr + Cf(Rp - Pr)}{A}.$$

COROLLARIUM 3

593. Si sit $C = 0$ et $A = 1$, erit $\Pi : z = \int dz = z - f$, quia integrale ita capi debet, ut evanescat posito $z = f$. Tum ergo erit $P = 1$, $Q = 1$ et $R = 1$, unde, ut sit $\Pi : s = \Pi : p + \Pi : q - \Pi : r$ seu $s = p + q - r$, oportet esse $s = -r + q + p$ et $\sqrt{(1 + 0ss)} = 1$, uti per se constat.

COROLLARIUM 4

594. Si sumatur $A = 1$ et $C = -1$ fiatque $\Pi : z = \text{Arc.}\cos.z$, ut sit $f = 1$, erit

$$\text{Arc.}\cos.s = \text{Arc.}\cos.p + \text{Arc.}\cos.q - \text{Arc.}\cos.r,$$

si fuerit

$$s = pqr - PQr + PRq + QRp$$

et

$$\sqrt{(1 - ss)} = PQR + Pqr + Qpr - Rpq,$$

unde sumto $r = 1$, ut sit $R = 0$ et $\text{Arc.}\cos.r = 0$, erit

$$s = pq - PQ \quad \text{et} \quad \sqrt{(1 - ss)} = Pq + Qp.$$

SCHOLION

595. Hinc notae regulae pro cosinibus deducuntur, quas fusius non prosequor. Verum casus facillimus, quo $A = 0$ et $C = 1$ hincque fit

$$\Pi : z = \int \frac{dz}{z} = lz$$

existente $f = 1$, insigni difficultate premi videtur ob expressiones pro s et $\sqrt{(A + Css)} = s$ in infinitum abeuntes. Cui incommodo ut occurratur, primo quidem numerus A ut infinite parvus spectetur eritque

$$P = \sqrt{(pp + A)} = p + \frac{A}{2p}, \quad Q = q + \frac{A}{2q}, \quad R = r + \frac{A}{2r}.$$

Quare ut fiat $ls = lp + lq - lr$, reperitur

$$As = -r\left(p + \frac{A}{2p}\right)\left(q + \frac{A}{2q}\right) - pqr + q\left(p + \frac{A}{2p}\right)\left(r + \frac{A}{2r}\right) + p\left(q + \frac{A}{2q}\right)\left(r + \frac{A}{2r}\right)$$

ac singulis membris evolutis

$$As = -\frac{Aqr}{2p} - \frac{Apr}{2q} + \frac{Aqr}{2p} + \frac{Apq}{2r} + \frac{Apr}{2q} + \frac{Apq}{2r}$$

seu $s = \frac{pq}{r}$, uti natura logarithmorum exigit.

Caeterum ex formulis inventis haud difficulter multiplicatio huiusmodi functionum transcendentium colligitur; veluti ut sit $\Pi : y = n\Pi : x$, relatio inter x et y algebraice assignari poterit.

PROBLEMA 76

596. *Si ponatur* $\Pi : z = \int \frac{dz(L + Mzz)}{\sqrt{(A + Czz)}}$ *sumto hoc integrali ita, ut evanescat posito* $z = 0$, *comparationem inter huiusmodi functiones transcendentes investigare.*

SOLUTIO

Statuatur inter binas variabiles x et y ista relatio

$$\alpha + \gamma(xx + yy) + 2\delta xy = 0,$$

unde fit

$$y = \frac{-\delta x + \sqrt{(-\alpha\gamma + (\delta\delta - \gamma\gamma)xx)}}{\gamma}.$$

Ponatur $-\alpha\gamma = Am$ et $\delta\delta - \gamma\gamma = Cm$, ut sit

$$\gamma y + \delta x = \sqrt{m(A + Cxx)} \quad \text{et} \quad \gamma x + \delta y = \sqrt{m(A + Cyy)}.$$

At illam aequationem differentiando fit

$$dx(\gamma x + \delta y) + dy(\gamma y + \delta x) = 0$$

seu

$$\frac{dx}{\sqrt{(A + Cxx)}} + \frac{dy}{\sqrt{(A + Cyy)}} = 0.$$

Iam statuatur

$$\frac{dx(L+Mxx)}{\sqrt{(A+Cxx)}}+\frac{dy(L+Myy)}{\sqrt{(A+Cyy)}}=d\,V\sqrt{m},$$

ut sit integrando

$$\Pi:x+\Pi:y=\text{Const.}+V\sqrt{m}.$$

Cum igitur sit $\frac{dy}{\sqrt{(A+Cyy)}}=\frac{-dx}{\sqrt{(A+Cxx)}}$, erit

$$d\,V\sqrt{m}=\frac{M\,dx(xx-yy)}{\sqrt{(A+Cxx)}}$$

hincque ob $y=\frac{\sqrt{m(A+Cxx)}-\delta x}{\gamma}$ erit

$$xx-yy=\frac{1}{\gamma\gamma}(\gamma\gamma xx-mA-mCxx-\delta\delta xx+2\delta x\sqrt{m(A+Cxx)}).$$

At $\gamma\gamma-\delta\delta=-mC$, ergo

$$d\,V\sqrt{m}=\frac{M\,dx(2\delta x\sqrt{m(A+Cxx)}-mA-2mCxx)}{\gamma\gamma\sqrt{(A+Cxx)}},$$

cuius integrale commode capi potest, dum fit

$$V\sqrt{m}=\frac{\delta Mxx\sqrt{m}}{\gamma\gamma}-\frac{Mmx}{\gamma\gamma}\sqrt{(A+Cxx)},$$

quae formula ob $\sqrt{m(A+Cxx)}=\gamma y+\delta x$ abit in

$$V\sqrt{m}=\frac{\delta Mxx-\gamma Mxy-\delta Mxx}{\gamma\gamma}\sqrt{m}=-\frac{Mxy}{\gamma}\sqrt{m}.$$

Quocirca habebimus

$$\Pi:x+\Pi:y=\text{Const.}-\frac{Mxy}{\gamma}\sqrt{m}$$

existente $\gamma y+\delta x=\sqrt{m(A+Cxx)}$ et $\gamma x+\delta y=\sqrt{m(A+Cyy)}$ ac praeterea $-\alpha\gamma=Am$ et $\delta\delta-\gamma\gamma=Cm$.

Ad constantem definiendam sumamus posito $x=0$ fieri $y=b$, ut sit

$$\Pi:x+\Pi:y=\Pi:b-\frac{Mxy}{\gamma}\sqrt{m}.$$

Tum vero est $\gamma b = \sqrt{mA}$ et $\delta b = \sqrt{(mA + mCbb)}$, ergo

$$\gamma = \frac{\sqrt{mA}}{b} \quad \text{et} \quad \delta = \frac{\sqrt{(mA + mCbb)}}{b}.$$

Hinc ergo concludimus, si fuerit

$$y\sqrt{A} + x\sqrt{(A + Cbb)} = b\sqrt{(A + Cxx)}$$

et, quod eodem redit,

$$x\sqrt{A} + y\sqrt{(A + Cbb)} = b\sqrt{(A + Cyy)},$$

fore

$$\Pi : x + \Pi : y = \Pi : b - \frac{Mbxy}{\sqrt{A}}$$

denotante Π eiusmodi functionem quantitatis suffixae, ut sit

$$\Pi : z = \int \frac{dz(L + Mzz)}{\sqrt{(A + Czz)}}$$

integrali hoc ita sumto, ut evanescat posito $z = 0$.

Natura harum functionum stabilita ac sublato discrimine inter quantitates constantes ac variabiles erit

$$\Pi : r = \Pi : p + \Pi : q + \frac{Mpqr}{\sqrt{A}},$$

si fuerit

$$q\sqrt{A} + p\sqrt{(A + Crr)} = r\sqrt{(A + Cpp)}$$

et

$$p\sqrt{A} + q\sqrt{(A + Crr)} = r\sqrt{(A + Cqq)},$$

unde fit

$$r = \frac{p\sqrt{(A + Cqq)} + q\sqrt{(A + Cpp)}}{\sqrt{A}}$$

et

$$\sqrt{(A + Crr)} = \frac{Cpq + \sqrt{(A + Cpp)(A + Cqq)}}{\sqrt{A}}.$$

COROLLARIUM 1

597. Sumto z negativo est $\Pi : -z = -\Pi : z$, unde capiendo quantitates p et q negative fiet

$$\Pi : p + \Pi : q + \Pi : r = \frac{Mpqr}{\sqrt{A}},$$

si fuerit

$$p\sqrt{A}+q\sqrt{(A+Crr)}+r\sqrt{(A+Cqq)}=0$$

seu

$$q\sqrt{A}+p\sqrt{(A+Crr)}+r\sqrt{(A+Cpp)}=0$$

seu

$$r\sqrt{A}+p\sqrt{(A+Cqq)}+q\sqrt{(A+Cpp)}=0$$

vel

$$Cpq-\sqrt{A(A+Crr)}+\sqrt{(A+Cpp)(A+Cqq)}=0,$$

ex qua formatur haec relatio

$$Cpqr+p\sqrt{(A+Cqq)(A+Crr)}+q\sqrt{(A+Cpp)(A+Crr)} \\ +r\sqrt{(A+Cpp)(A+Cqq)}=0.$$

COROLLARIUM 2

598. Hac ergo methodo tres huiusmodi functiones $\Pi:z$ exhiberi possunt, quarum summam algebraice exprimere licet; quod autem de summa ostendimus, valet quoque de summa binarum demta tertia.

COROLLARIUM 3

599. Si ponamus $L=A$ et $M=C$, functio proposita

$$\Pi:z=\int dz\sqrt{(A+Czz)}$$

exprimit arcum curvae, cuius abscissae z convenit applicata $\sqrt{(A+Czz)}$, et summa trium huiusmodi arcuum ita algebraice dabitur

$$\Pi:p+\Pi:q+\Pi:r=\frac{Cpqr}{\sqrt{A}},$$

si inter p, q, r superior relatio statuatur.

SCHOLION

600. Haec proprietas inde est nata, quod differentiale dV integrationem admisit. Cum nempe esset

$$dV\sqrt{m}=\frac{Mdx(xx-yy)}{\sqrt{(A+Cxx)}},$$

ob $\sqrt{m(A + Cxx)} = \gamma y + \delta x$ erit

$$dV = \frac{Mdx(xx - yy)}{\gamma y + \delta x},$$

cuius integrale commode ex aequatione assumta $\alpha + \gamma(xx + yy) + 2\delta xy = 0$ definiri potest. Ponatur enim

$$xx + yy = tt \quad \text{et} \quad xy = u;$$

erit

$$\alpha + \gamma tt + 2\delta u = 0$$

et differentialibus sumendis

$$xdx + ydy = tdt, \quad xdy + ydx = du \quad \text{et} \quad \gamma tdt + \delta du = 0;$$

ex binis prioribus colligitur

$$(xx - yy)dx = xtdt - ydu$$

et ob $tdt = -\frac{\delta du}{\gamma}$ erit

$$(xx - yy)dx = -\frac{du}{\gamma}(\delta x + \gamma y),$$

ita ut sit

$$\frac{dx(xx - yy)}{\gamma y + \delta x} = -\frac{du}{\gamma}$$

hincque $dV = -\frac{Mdu}{\gamma}$, unde manifesto sequitur $V = -\frac{Mu}{\gamma} = -\frac{Mxy}{\gamma}$, uti in solutione operosius eruimus. Verum hac operatione commode uti licebit in sequente problemate, ubi formulas magis complexas sumus contemplaturi.

PROBLEMA 77

601. *Si ponatur*

$$\Pi : z = \int \frac{dz(L + Mz^2 + Nz^4 + Oz^6)}{\sqrt{(A + Czz)}}$$

integrali hoc ita sumto, ut evanescat posito $z = 0$, *comparationem inter huiusmodi functiones transcendentes investigare.*

SOLUTIO

Posita ut ante inter variabiles x et y hac relatione

$$\alpha + \gamma(xx + yy) + 2\delta xy = 0$$

sit $-\alpha\gamma = Am$ et $\delta\delta - \gamma\gamma = Cm$ fietque

$$\gamma y + \delta x = \sqrt{m(A + Cxx)} \quad \text{et} \quad \gamma x + \delta y = \sqrt{m(A + Cyy)}$$

sumtisque differentialibus

$$\frac{dx}{\sqrt{(A+Cxx)}} + \frac{dy}{\sqrt{(A+Cyy)}} = 0.$$

Iam statuatur

$$\frac{dx(L+Mxx+Nx^4+Ox^6)}{\sqrt{(A+Cxx)}} + \frac{dy(L+Myy+Ny^4+Oy^6)}{\sqrt{(A+Cyy)}} = dV\sqrt{m},$$

ut sit

$$\Pi : x + \Pi : y = \text{Const.} + V\sqrt{m}.$$

At ob $\frac{dy}{\sqrt{(A+Cyy)}} = -\frac{dx}{\sqrt{(A+Cxx)}}$ ista aequatio abit in

$$\frac{dx(M(xx-yy)+N(x^4-y^4)+O(x^6-y^6))}{\sqrt{(A+Cxx)}} = dV\sqrt{m}$$

et ob $\sqrt{m(A+Cxx)} = \gamma y + \delta x$ in hanc

$$\frac{dx(xx-yy)(M+N(xx+yy)+O(x^4+xxyy+y^4))}{\gamma y + \delta x} = dV.$$

Sit nunc $xx + yy = tt$ et $xy = u$, ut habeatur

$$\alpha + \gamma tt + 2\delta u = 0 \quad \text{et} \quad \gamma t dt + \delta du = 0 \quad \text{seu} \quad tdt = -\frac{\delta du}{\gamma},$$

atque ob $xdx + ydy = tdt$ et $xdy + ydx = du$ hinc colligimus

$$(xx-yy)dx = xtdt - ydu = -\frac{du}{\gamma}(\gamma y + \delta x)$$

ideoque

$$\frac{dx(xx-yy)}{\gamma y+\delta x} = -\frac{du}{\gamma},$$

unde habebimus

$$dV = -\frac{du}{\gamma}\left(M + N(xx+yy) + O(x^4+xxyy+y^4)\right).$$

At est

$$xx + yy = tt = \frac{-\alpha - 2\delta u}{\gamma} \quad \text{et} \quad x^4 + xxyy + y^4 = t^4 - uu.$$

Notetur autem esse $\frac{du}{\gamma} = -\frac{tdt}{\delta}$, unde concludimus

$$dV = -\frac{Mdu}{\gamma} + \frac{Nt^3dt}{\delta} + \frac{Ot^5dt}{\delta} + \frac{Ouudu}{\gamma},$$

sicque prodit integrando

$$V = -\frac{Mu}{\gamma} + \frac{Nt^4}{4\delta} + \frac{Ot^6}{6\delta} + \frac{Ou^3}{3\gamma}.$$

Quodsi iam ponamus fieri $y = b$, si $x = 0$, erit

$$\gamma = \frac{\sqrt{mA}}{b}, \quad \delta = \frac{\sqrt{m(A+Cbb)}}{b} \quad \text{et} \quad \alpha = -b\sqrt{mA},$$

tum vero

$$y\sqrt{A} + x\sqrt{(A+Cbb)} = b\sqrt{(A+Cxx)},$$

$$x\sqrt{A} + y\sqrt{(A+Cbb)} = b\sqrt{(A+Cyy)}$$

et

$$b\sqrt{A} = x\sqrt{(A+Cyy)} + y\sqrt{(A+Cxx)}.$$

Hinc, cum sit

$$V = -\frac{Mbxy}{\sqrt{mA}} + \frac{Nb(xx+yy)^2}{4\sqrt{m(A+Cbb)}} + \frac{Ob(xx+yy)^3}{6\sqrt{m(A+Cbb)}} + \frac{Obx^3y^3}{3\sqrt{mA}},$$

nostra relatio, cui satisfaciunt praecedentes determinationes, inter functiones transcendentes erit

$$\Pi:x + \Pi:y = \Pi:b - \frac{Mbxy}{\sqrt{A}} + \frac{Nb(xx+yy)^2}{4\sqrt{(A+Cbb)}} + \frac{Ob(xx+yy)^3}{6\sqrt{(A+Cbb)}}$$
$$+ \frac{Obx^3y^3}{3\sqrt{A}} - \frac{Nb^5}{4\sqrt{(A+Cbb)}} - \frac{Ob^7}{6\sqrt{(A+Cbb)}},$$

ubi notandum est esse in rationalibus

$$-b\sqrt{A} + \frac{(xx+yy)\sqrt{A}}{b} + \frac{2xy\sqrt{(A+Cbb)}}{b} = 0$$

seu

$$xx + yy = bb - \frac{2xy\sqrt{(A+Cbb)}}{\sqrt{A}}.$$

Hinc colligitur

$$(xx+yy)^2 - b^4 = -\frac{4bbxy\sqrt{(A+Cbb)}}{\sqrt{A}} + \frac{4xxyy(A+Cbb)}{A}$$

et

$$(xx+yy)^3 - b^6 = -\frac{6b^4xy\sqrt{(A+Cbb)}}{\sqrt{A}} + \frac{12bbxxyy(A+Cbb)}{A} - \frac{8x^3y^3(A+Cbb)^{\frac{3}{2}}}{A\sqrt{A}},$$

ita ut nostra aequatio sit

$$\Pi : x + \Pi : y = \Pi : b - \frac{Mbxy}{\sqrt{A}} - \frac{Nb^3xy}{\sqrt{A}} + \frac{Nbxxyy}{A}\sqrt{(A+Cbb)} - \frac{Ob^5xy}{\sqrt{A}}$$
$$+ \frac{2\,Ob^3xxyy}{A}\sqrt{(A+Cbb)} - \frac{Obx^3y^3}{3A\sqrt{A}}(3A+4Cbb).$$

COROLLARIUM 1

602. Si ponamus $b = r$, $x = -p$, $y = -q$, erit nostra aequatio

$$\Pi : p + \Pi : q + \Pi : r = \frac{pqr}{\sqrt{A}}(M + Nrr + Or^4) - \frac{ppqq\sqrt{(A+Crr)}}{A}(Nr + 2\,Or^3)$$
$$+ \frac{Op^3q^3r}{3A\sqrt{A}}(3A + 4Crr)$$

existente

$$pp + qq = rr - \frac{2pq}{\sqrt{A}}\sqrt{(A+Crr)},$$

unde fit

$$\frac{\sqrt{(A+Crr)}}{\sqrt{A}} = \frac{rr-pp-qq}{2pq}.$$

COROLLARIUM 2

603. Substituto hoc valore pro $\frac{\sqrt{(A+Crr)}}{\sqrt{A}}$ sequens obtinebitur aequatio, in quam ternae quantitates p, q, r aequaliter ingrediuntur,

$$\Pi : p + \Pi : q + \Pi : r = \frac{Mpqr}{\sqrt{A}} + \frac{Npqr}{2\sqrt{A}}(pp + qq + rr)$$
$$+ \frac{Opqr}{3\sqrt{A}}(p^4 + q^4 + r^4 + ppqq + pprr + qqrr),$$

cui satisfaciunt formulae supra (§ 602) datae vel haec rationalis

$$\frac{4\,Cppqqrr}{A} = p^4 + q^4 + r^4 - 2ppqq - 2pprr - 2qqrr.$$

COROLLARIUM 3

604. Si numeratori formulae integralis adhuc adiecissemus terminum Pz^8, ut esset

$$\Pi : z = \int \frac{dz(L + Mz^2 + Nz^4 + Oz^6 + Pz^8)}{\sqrt{(A+Czz)}},$$

ad aequationem modo inventam adhuc accessisset terminus

$$\frac{Ppqr}{4\sqrt{A}}\left(p^6+q^6+r^6+ppq^4+ppr^4+p^4qq+p^4rr+q^4rr+qqr^4+\frac{4}{3}ppqqrr\right).$$

SCHOLION

605. Istae relationes quoque ex superioribus reductionibus[1]) derivari possunt; cum enim inde sit $\Pi : z = E\int\frac{dz}{\sqrt{(A+Czz)}} +$ *quantitate algebraica*, si hic pro z successive quantitates p, q, r substituamus ita a se invicem pendentes, ut ante declaravimus, erit

$$\int\frac{dp}{\sqrt{(A+Cpp)}}+\int\frac{dq}{\sqrt{(A+Cqq)}}+\int\frac{dr}{\sqrt{(A+Crr)}}=0,$$

unde concludimus

$$\Pi : p + \Pi : q + \Pi : r = f : p + f : q + f : r$$

denotante f functionem quandam algebraicam quantitatis suffixae; atque summa harum trium functionum rediret ad expressionem ante inventam, si modo relationis inter p, q, r datae ratio habeatur, scilicet inde littera C eliminari deberet. Haec autem reductio ingentem laborem requireret. Hic vero imprimis methodum, qua hic sum usus, spectari convenit, quae cum sit prorsus singularis, ad magis arduam deducere videtur. Certe comparatio functionum transcendentium, quam in capite sequente sum traditurus, vix alia methodo investigari posse videtur, unde huius methodi utilitas in sequenti capite potissimum cernetur.

1) Vide formulam § 111 evolutam et praeterea § 113, p. 64 et 65. F. E.

CAPUT VI

DE COMPARATIONE QUANTITATUM TRANSCENDENTIUM IN FORMA $\int \frac{P\,dz}{\sqrt{(A + 2Bz + Czz + 2Dz^3 + Ez^4)}}$ CONTENTARUM[1])

PROBLEMA 78

606. *Proposita relatione inter x et y hac*

$$\alpha + \gamma(xx + yy) + 2\delta xy + \zeta xxyy = 0$$

inde elicere functiones transcendentes formae praescriptae, quas inter se comparare liceat.

SOLUTIO

Ex proposita aequatione definiatur utraque variabilis

$$y = \frac{-\delta x + \sqrt{(-\alpha\gamma + (\delta\delta - \gamma\gamma - \alpha\zeta)xx - \gamma\zeta x^4)}}{\gamma + \zeta xx}$$

et

$$x = \frac{-\delta y + \sqrt{(-\alpha\gamma + (\delta\delta - \gamma\gamma - \alpha\zeta)yy - \gamma\zeta y^4)}}{\gamma + \zeta yy},$$

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1) Vide L. EULERI Commentationem 261 (indicis ENESTROEMIANI): *Specimen alterum methodi novae quantitates transcendentes inter se comparandi*, Novi comment. acad. sc. Petrop. 7 (1758/9), 1761, p. 3; *LEONHARDI EULERI Opera omnia*, series I, vol. 20, p. 153; Commentationem 818: *De comparatione arcuum curvarum irrectificabilium*, Opera postuma **1**, Petropoli 1862, p. 452; *LEONHARDI EULERI Opera omnia*, series I, vol. 21, p. 296; Commentationem 347: *Evolutio generalior formularum comparationi curvarum inservientium*, Novi comment. acad. sc. Petrop. **12** (1766/7), 1768, p. 42; *LEONHARDI EULERI Opera omnia*, series I, vol. 20, p. 318. Vide etiam Commentationes 582 et 676, *LEONHARDI EULERI Opera omnia*, series I, vol. 21, p. 57 et 207. F. E.

quae radicalia ad formam praescriptam revocentur ponendo

$$-\alpha\gamma = Am, \quad \delta\delta - \gamma\gamma - \alpha\zeta = Cm \quad \text{et} \quad -\gamma\zeta = Em,$$

unde fit

$$\alpha = -\frac{Am}{\gamma}, \quad \zeta = -\frac{Em}{\gamma} \quad \text{et} \quad \delta\delta = Cm + \gamma\gamma + \frac{AEmm}{\gamma\gamma}.$$

Erit ergo

$$\gamma y + \delta x + \zeta xxy = \sqrt{m(A + Cxx + Ex^4)},$$

$$\gamma x + \delta y + \zeta xyy = \sqrt{m(A + Cyy + Ey^4)}.$$

Ipsa autem aequatio proposita, si differentietur, dat

$$dx(\gamma x + \delta y + \zeta xyy) + dy(\gamma y + \delta x + \zeta xxy) = 0,$$

ubi illi valores substituti praebent

$$\frac{dx}{\sqrt{(A + Cxx + Ex^4)}} + \frac{dy}{\sqrt{(A + Cyy + Ey^4)}} = 0.$$

Vicissim ergo proposita hac aequatione differentiali ei satisfaciet haec aequatio finita

$$-Am + \gamma\gamma(xx + yy) + 2xy\sqrt{(\gamma^4 + Cm\gamma\gamma + AEmm)} - Emxxyy = 0$$

seu ponendo $\frac{\gamma\gamma}{m} = k$ haec

$$-A + k(xx + yy) + 2xy\sqrt{(kk + kC + AE)} - Exxyy = 0,$$

quae cum involvat constantem k in aequatione differentiali non contentam, simul erit integrale completum.

Hinc autem fit

$$ky + x\sqrt{(kk + kC + AE)} - Exxy = \sqrt{k(A + Cxx + Ex^4)}$$

et

$$kx + y\sqrt{(kk + kC + AE)} - Exyy = \sqrt{k(A + Cyy + Ey^4)}.$$

COROLLARIUM 1

607. Constans k ita assumi potest, ut posito $x = 0$ fiat $y = b$; oritur autem

$$bk = \sqrt{Ak} \quad \text{et} \quad b\sqrt{(kk + kC + AE)} = \sqrt{k(A + Cbb + Eb^4)},$$

ergo

$$k = \frac{A}{bb} \quad \text{et} \quad \sqrt{(kk + kC + AE)} = \frac{1}{bb}\sqrt{A(A + Cbb + Eb^4)}$$

ideoque habebimus

$$Ay + x\sqrt{A(A + Cbb + Eb^4)} - Ebbxxy = b\sqrt{A(A + Cxx + Ex^4)}$$

et

$$Ax + y\sqrt{A(A + Cbb + Eb^4)} - Ebbxyy = b\sqrt{A(A + Cyy + Ey^4)}.$$

COROLLARIUM 2

608. Haec igitur relatio finita inter x et y erit integrale completum aequationis differentialis

$$\frac{dx}{\sqrt{(A + Cxx + Ex^4)}} + \frac{dy}{\sqrt{(A + Cyy + Ey^4)}} = 0,$$

quod rationaliter inter x et y expressum erit

$$A(xx + yy - bb) + 2xy\sqrt{A(A + Cbb + Eb^4)} - Ebbxxyy = 0.$$

COROLLARIUM 3

609. Hinc ergo y ita per x exprimetur, ut sit

$$y = \frac{b\sqrt{A(A + Cxx + Ex^4)} - x\sqrt{A(A + Cbb + Eb^4)}}{A - Ebbxx},$$

atque ex hoc valore elicitur

$$\sqrt{\frac{A + Cyy + Ey^4}{A}}$$

$$= \frac{(A + Ebbxx)\sqrt{(A + Cbb + Eb^4)(A + Cxx + Ex^4)} - 2AEbx(bb + xx) - Cbx(A + Ebbxx)}{(A - Ebbxx)^2}.$$

COROLLARIUM 4

610. Hinc constantem b pro lubitu determinando infinita integralia particularia exhiberi possunt, quorum praecipua sunt:

1) sumendo $b = 0$, unde fit $y = -x$;

2) sumendo $b = \infty$, unde fit $y = \frac{\sqrt{A}}{x\sqrt{E}}$;

3) si $A + Cbb + Eb^4 = 0$ hincque $bb = \frac{-C + \sqrt{(CC - 4AE)}}{2E}$, unde fit $y = \frac{b\sqrt{A(A + Cxx + Ex^4)}}{A - Ebbxx}$.

SCHOLION

611. Hic iam usus istius methodi, qua retrogrediendo ab aequatione finita ad aequationem differentialem pervenimus, luculenter perspicitur. Cum enim integratio formulae $\frac{dx}{\sqrt{(A + Cxx + Ex^4)}}$ nullo modo neque per logarithmos neque arcus circulares perfici posset, mirum sane est talem aequationem differentialem adeo algebraice integrari posse; quae quidem in praecedente capite ope eiusdem methodi sunt tradita, etiam methodo ordinaria erui possunt, dum singulae formulae differentiales vel per logarithmos vel arcus circulares exprimuntur, quorum deinceps comparatio ad aequationem algebraicam reducitur. Verum quia hic talis integratio plane non locum invenit, nulla certe alia methodus patet, qua idem integrale, quod hic exhibuimus, investigari posset. Quare hoc argumentum diligentius evolvamus.

PROBLEMA 79

612. *Si $\Pi : z$ denotet eiusmodi functionem ipsius z, ut sit*

$$\Pi : z = \int \frac{dz}{\sqrt{(A + Czz + Ez^4)}}$$

integrali ita sumto, ut evanescat posito $z = 0$, comparationem inter huiusmodi functiones investigare.

SOLUTIO

Posita inter binas variabiles x et y relatione supra definita vidimus fore

$$\frac{dx}{\sqrt{(A + Cxx + Ex^4)}} + \frac{dy}{\sqrt{(A + Cyy + Ey^4)}} = 0.$$

Hinc, cum posito $x = 0$ fiat $y = b$, elicitur integrando

$$\Pi : x + \Pi : y = \Pi : b.$$

Cum iam nullum amplius discrimen inter variabiles x, y et constantem b intercedat, statuamus $x=p$, $y=q$ et $b=-r$, ut sit $\Pi:b=-\Pi:r$, atque haec relatio inter functiones transcendentes

$$\Pi:p+\Pi:q+\Pi:r=0$$

per sequentes formulas algebraicas exprimetur

$$(A-Epprr)q+p\sqrt{A(A+Crr+Er^4)}+r\sqrt{A(A+Cpp+Ep^4)}=0$$

seu

$$(A-Eppqq)r+q\sqrt{A(A+Cpp+Ep^4)}+p\sqrt{A(A+Cqq+Eq^4)}=0$$

seu

$$(A-Eqqrr)p+r\sqrt{A(A+Cqq+Eq^4)}+q\sqrt{A(A+Crr+Er^4)}=0,$$

quae oriuntur ex hac aequatione

$$A(pp+qq-rr)-Eppqqrr+2pq\sqrt{A(A+Crr+Er^4)}=0.$$

Haec vero ad rationalitatem perducta fit

$$AA(p^4+q^4+r^4-2ppqq-2pprr-2qqrr)-2AEppqqrr(pp+qq+rr)$$
$$-4ACppqqrr+EEp^4q^4r^4=0,$$

quae autem ob pluralitatem radicum satisfacit omnibus signorum variationibus in superiori aequatione transcendente.

COROLLARIUM 1

613. Sumamus r negative, ut fiat

$$\Pi:r=\Pi:p+\Pi:q,$$

eritque

$$r=\frac{p\sqrt{A(A+Cqq+Eq^4)}+q\sqrt{A(A+Cpp+Ep^4)}}{A-Eppqq},$$

unde colligitur fore

$$\sqrt{\frac{A+Crr+Er^4}{A}}$$
$$=\frac{(A+Eppqq)\sqrt{(A+Cpp+Ep^4)(A+Cqq+Eq^4)}+2AEpq(pp+qq)+Cpq(A+Eppqq)}{(A-Eppqq)^2}.$$

COROLLARIUM 2

614. Quodsi ergo ponamus $q=p$, ut sit

$$\Pi : r = 2\Pi : p,$$

erit

$$r = \frac{2p\sqrt{A(A+Cpp+Ep^4)}}{A-Ep^4}$$

atque

$$\sqrt{\frac{A+Crr+Er^4}{A}} = \frac{AA+2ACpp+6AEp^4+2CEp^6+EEp^8}{(A-Ep^4)^2}.$$

Hoc igitur modo functio assignari potest aequalis duplo similis functionis.

COROLLARIUM 3

615. Si ponatur

$$q = \frac{2p\sqrt{A(A+Cpp+Ep^4)}}{A-Ep^4}$$

et

$$\sqrt{A(A+Cqq+Eq^4)} = \frac{A(AA+2ACpp+6AEp^4+2CEp^6+EEp^8)}{(A-Ep^4)^2},$$

ut sit $\Pi : q = 2\Pi : p$, fiet ex primo corollario

$$\Pi : r = 3\Pi : p.$$

Tum igitur erit

$$r = \frac{p(3AA+4ACpp+6AEp^4-EEp^8)}{AA-6AEp^4-4CEp^6-3EEp^8}.$$

SCHOLION 1

616. Nimis operosum est hanc functionum multiplicationem ulterius continuare multoque minus legem in earum progressione deprehendere licet. Quodsi ponamus brevitatis gratia

$$\sqrt{A(A+Cpp+Ep^4)} = AP \quad \text{et} \quad A-Ep^4 = A\mathfrak{P},$$

ut sit

$$Cpp = APP - A - Ep^4 \quad \text{et} \quad Ep^4 = A(1-\mathfrak{P}),$$

hae multiplicationes usque ad quadruplum ita se habebunt; scilicet si statuamus

$$\Pi : r = 2\Pi : p, \quad \Pi : s = 3\Pi : p \quad \text{et} \quad \Pi : t = 4\Pi : p,$$

reperietur

$$r=\frac{2Pp}{\mathfrak{P}},\quad s=\frac{p(4PP-\mathfrak{P}\mathfrak{P})}{\mathfrak{P}\mathfrak{P}-4PP(1-\mathfrak{P})},\quad t=\frac{4pP\mathfrak{P}(2PP(2-\mathfrak{P})-\mathfrak{P}\mathfrak{P})}{\mathfrak{P}^4-16P^4(1-\mathfrak{P})}.$$

Quodsi simili modo ponamus

$$\sqrt{A(A+Crr+Er^4)}=AR\quad\text{et}\quad A-Er^4=A\mathfrak{R},$$

erit

$$R=\frac{2PP(2-\mathfrak{P})-\mathfrak{P}\mathfrak{P}}{\mathfrak{P}\mathfrak{P}}\quad\text{et}\quad\mathfrak{R}=\frac{\mathfrak{P}^4-16P^4(1-\mathfrak{P})}{\mathfrak{P}^4},$$

unde pro quadruplicatione fit

$$t=\frac{2Rr}{\mathfrak{R}},\quad T=\frac{2RR(2-\mathfrak{R})-\mathfrak{R}\mathfrak{R}}{\mathfrak{R}\mathfrak{R}},\quad\mathfrak{T}=\frac{\mathfrak{R}^4-16R^4(1-\mathfrak{R})}{\mathfrak{R}^4}.$$

Quare si pro octuplicatione statuamus $\Pi:z=8\Pi:p$, erit

$$z=\frac{2Tt}{\mathfrak{T}}=\frac{4rR\mathfrak{R}(2RR(2-\mathfrak{R})-\mathfrak{R}\mathfrak{R})}{\mathfrak{R}^4-16R^4(1-\mathfrak{R})}.$$

Hinc intelligitur, quomodo in continua duplicatione versari oporteat, neque tamen legem progressionis observare licet. Caeterum cognitio huius legis ad incrementum Analyseos maxime esset optanda, ut inde generatim relatio inter z et p pro aequalitate $\Pi:z=n\Pi:p$ definiri posset, quemadmodum hoc in capite praecedente successit; hinc enim eximias proprietates circa integralia formae $\int\frac{dz}{\sqrt{(A+Czz+Ez^4)}}$ cognoscere liceret, quibus scientia analytica haud mediocriter promoveretur.

SCHOLION 2

617. Modus maxime idoneus in legem progressionis inquirendi videtur, si ternos terminos se ordine excipientes contemplemur hoc modo

$$\Pi:x=(n-1)\Pi:p,\quad\Pi:y=n\Pi:p,\quad\Pi:z=(n+1)\Pi:p;$$

ubi cum sit

$$\Pi:x=\Pi:y-\Pi:p\quad\text{et}\quad\Pi:z=\Pi:y+\Pi:p,$$

erit

$$x=\frac{y\sqrt{A(A+Cpp+Ep^4)}-p\sqrt{A(A+Cyy+Ey^4)}}{A-Eppyy},$$

$$z=\frac{y\sqrt{A(A+Cpp+Ep^4)}+p\sqrt{A(A+Cyy+Ey^4)}}{A-Eppyy},$$

unde concludimus

$$(A - Eppyy)(x + z) = 2y\sqrt{A(A + Cpp + Ep^4)}.$$

Ponamus ut ante $\sqrt{A(A + Cpp + Ep^4)} = AP$ et $A - Ep^4 = A\mathfrak{P}$, et quia singulae quantitates x, y, z factorem p simpliciter involvunt, sit

$$x = pX, \quad y = pY \quad \text{et} \quad z = pZ;$$

erit

$$(1 - (1 - \mathfrak{P})YY)(X + Z) = 2PY \quad \text{seu} \quad Z = \frac{2PY}{1 - (1 - \mathfrak{P})YY} - X,$$

cuius formulae ope ex binis terminis contiguis X et Y sequens Z haud difficulter invenitur. Quod quo facilius appareat, ponatur $2P = Q$ et $1 - \mathfrak{P} = \mathfrak{Q}$, ut sit $Z = \frac{QY}{1 - \mathfrak{Q}YY} - X$. Iam progressio quaesita ita se habebit

$$1)\ 1, \quad 2)\ \frac{Q}{\mathfrak{P}}, \quad 3)\ \frac{QQ - \mathfrak{P}\mathfrak{P}}{\mathfrak{P}\mathfrak{P} - QQ\mathfrak{Q}}, \quad 4)\ \frac{Q^3\mathfrak{P}(1 + \mathfrak{Q}) - 2Q\mathfrak{P}^3}{\mathfrak{P}^4 - Q^4\mathfrak{Q}},$$

$$5)\ \frac{\mathfrak{P}^6 - 3QQ\mathfrak{P}^4 + Q^4\mathfrak{P}\mathfrak{P}(1 + 2\mathfrak{Q}) - Q^6\mathfrak{Q}\mathfrak{Q}}{\mathfrak{P}^6 - 3QQ\mathfrak{P}^4\mathfrak{Q} + Q^4\mathfrak{P}\mathfrak{P}\mathfrak{Q}(2 + \mathfrak{Q}) - Q^6\mathfrak{Q}} \quad \text{etc.}$$

Quaestio ergo huc redit, ut investigetur progressio ex data relatione inter ternos terminos successivos X, Y, Z, quae sit $Z = \frac{QY}{1 - \mathfrak{Q}YY} - X$, existente termino primo $= 1$ et secundo $= \frac{Q}{1 - \mathfrak{Q}}$.

PROBLEMA 80

618. *Si $\Pi : z$ eiusmodi denotet functionem ipsius z, ut sit*

$$\Pi : z = \int \frac{dz(L + Mzz + Nz^4)}{\sqrt{(A + Czz + Ez^4)}}$$

integrali ita sumto, ut evanescat posito $z = 0$, comparationem inter huiusmodi functiones transcendentes investigare.

SOLUTIO

Stabilita inter binas variabiles x et y hac relatione, ut sit

$$Ay + \mathfrak{B}x - Ebbxxy = b\sqrt{A(A + Cxx + Ex^4)}$$

seu

$$Ax + \mathfrak{B}y - Ebbxyy = b\sqrt{A(A + Cyy + Ey^4)}$$

sive sublata irrationalitate

$$A(xx + yy - bb) + 2\mathfrak{B}xy - Ebbxxyy = 0$$

existente brevitatis gratia $\mathfrak{B} = \sqrt{A(A + Cbb + Eb^4)}$, erit, uti ante vidimus,

$$\frac{dx}{\sqrt{(A+Cxx+Ex^4)}} + \frac{dy}{\sqrt{(A+Cyy+Ey^4)}} = 0.$$

Ponamus igitur

$$\frac{dx(L+Mxx+Nx^4)}{\sqrt{(A+Cxx+Ex^4)}} + \frac{dy(L+Myy+Ny^4)}{\sqrt{(A+Cyy+Ey^4)}} = b\,dV\sqrt{A},$$

ut sit nostro signandi more

$$\Pi : x + \Pi : y = \text{Const.} + bV\sqrt{A},$$

ubi constans ita definiri debet, ut posito $x = 0$ fiat $y = b$.

Quaestio ergo ad inventionem functionis V revocatur; quem in finem loco dy valore ex priori aequatione substituto erit

$$b\,dV\sqrt{A} = \frac{dx(M(xx-yy) + N(x^4-y^4))}{\sqrt{(A+Cxx+Ex^4)}},$$

verum quia

$$b\sqrt{A(A + Cxx + Ex^4)} = Ay + \mathfrak{B}x - Ebbxxy,$$

habebimus

$$dV = \frac{dx(xx-yy)(M+N(xx+yy))}{Ay+\mathfrak{B}x-Ebbxxy}.$$

Sumamus iam aequationem rationalem

$$A(xx + yy - bb) + 2\mathfrak{B}xy - Ebbxxyy = 0$$

et ponamus $xx + yy = tt$ et $xy = u$, ut sit

$$A(tt - bb) + 2\mathfrak{B}u - Ebbuu = 0$$

ideoque

$$Atdt = -\mathfrak{B}du + Ebbudu.$$

Cum porro sit $xdx + ydy = tdt$ et $xdy + ydx = du$, erit

$$(xx - yy)dx = xtdt - ydu$$

seu

$$A(xx - yy)dx = -du(Ay + \mathfrak{B}x - Ebbxxy),$$

ita ut sit

$$\frac{dx(xx-yy)}{Ay + \mathfrak{B}x - Ebbxxy} = -\frac{du}{A},$$

ex quo deducitur

$$dV = -\frac{du}{A}(M + Ntt),$$

et ob $tt = bb - \frac{2\mathfrak{B}u}{A} + \frac{Ebbuu}{A}$ erit

$$dV = -\frac{du}{AA}(AM + ANbb - 2\mathfrak{B}Nu + ENbbuu),$$

unde integrando elicitur

$$V = -\frac{Mu}{A} - \frac{Nbbu}{A} + \frac{\mathfrak{B}Nuu}{AA} - \frac{ENbbu^3}{3AA}.$$

Hoc ergo valore substituto ob $u = xy$ habebimus

$$\Pi : x + \Pi : y = \Pi : b - \frac{Mbxy}{\sqrt{A}} - \frac{Nb^3xy}{\sqrt{A}} + \frac{\mathfrak{B}Nbx^2y^2}{A\sqrt{A}} - \frac{ENb^3x^3y^3}{3A\sqrt{A}}.$$

Cum autem sit

$$\mathfrak{B}xy = \frac{1}{2}Abb - \frac{1}{2}A(xx + yy) + \frac{1}{2}Ebbxxyy,$$

erit

$$\Pi : x + \Pi : y = \Pi : b - \frac{Mbxy}{\sqrt{A}} - \frac{Nbxy}{2A\sqrt{A}}\left(A(bb + xx + yy) - \frac{1}{3}Ebbxxyy\right),$$

cui ergo aequationi satisfit per formulas algebraicas supra exhibitas, quibus relatio inter x, y et b exprimitur. Quodsi ergo statuatur haec aequatio

$$\Pi : p + \Pi : q + \Pi : r = \frac{Mpqr}{\sqrt{A}} + \frac{Npqr}{2A\sqrt{A}}\left(A(pp + qq + rr) - \frac{1}{3}Eppqqrr\right),$$

ea efficitur sequenti relatione inter p, q, r constituta

$$(A - Eppqq)r + p\sqrt{A(A + Cqq + Eq^4)} + q\sqrt{A(A + Cpp + Ep^4)} = 0$$

seu

$$(A - Epprr)q + p\sqrt{A(A + Crr + Er^4)} + r\sqrt{A(A + Cpp + Ep^4)} = 0$$

seu

$$(A - Eqqrr)p + q\sqrt{A(A + Crr + Er^4)} + r\sqrt{A(A + Cqq + Eq^4)} = 0$$

sive per simplicem irrationalitatem

$$A(pp+qq-rr)+2pq\sqrt{A(A+Crr+Er^4)}-Eppqqrr=0$$

seu

$$A(pp+rr-qq)+2pr\sqrt{A(A+Cqq+Eq^4)}-Eppqqrr=0$$

seu

$$A(qq+rr-pp)+2qr\sqrt{A(A+Cpp+Ep^4)}-Eppqqrr=0$$

penitusque irrationalitate sublata

$$EEp^4q^4r^4-2AEppqqrr(pp+qq+rr)-4ACppqqrr$$
$$+AA(p^4+q^4+r^4-2ppqq-2pprr-2qqrr)=0.$$

COROLLARIUM 1

619. Sit $q=r=s$, ut habeamus hanc aequationem

$$\Pi:p+2\Pi:s=\frac{Mpss}{\sqrt{A}}+\frac{Npss}{2A\sqrt{A}}\left(A(pp+2ss)-\frac{1}{3}Epps^4\right),$$

cui satisfacit haec relatio

$$(A-Es^4)p+2s\sqrt{A(A+Css+Es^4)}=0.$$

COROLLARIUM 2

620. Sumamus s negative et loco p substituamus ibi hunc valorem, ut habeamus

$$2\Pi:s+\Pi:q+\Pi:r+\frac{Mpss}{\sqrt{A}}+\frac{Npss}{2A\sqrt{A}}\left(A(pp+2ss)-\frac{1}{3}Epps^4\right)$$
$$=\frac{Mpqr}{\sqrt{A}}+\frac{Npqr}{2A\sqrt{A}}\left(A(pp+qq+rr)-\frac{1}{3}Eppqqrr\right)$$

existente

$$p=\frac{2s\sqrt{A(A+Css+Es^4)}}{A-Es^4},$$

unde fit [§ 615]

$$\sqrt{A(A+Cpp+Ep^4)}=\frac{A(A+Css+Es^4)^2+A(4AE-CC)s^4}{(A-Es^4)^2},$$

qui valores in superioribus formulis substitui debent.

COROLLARIUM 3

621. Hoc modo effici poterit, ut partes algebraicae evanescant atque functiones transcendentes solae inter se comparentur. Veluti si esset $N=0$, statui oporteret $ss=qr$, ut fieret

$$2\Pi:s+\Pi:q+\Pi:r=0.$$

At posito $ss=qr$ fit

$$p=\frac{2\sqrt{Aqr(A+Cqr+Eqqrr)}}{A-Eqqrr}.$$

Est vero etiam

$$p=\frac{-q\sqrt{A(A+Crr+Er^4)}-r\sqrt{A(A+Cqq+Eq^4)}}{A-Eqqrr},$$

quibus valoribus aequatis oritur haec aequatio

$$(AA+EEq^4r^4)(qq-6qr+rr)-8Cqqrr(A+Eqqrr)$$
$$-2AEqqrr(qq+10qr+rr)=0.$$

SCHOLION

622. Si $\Pi:z$ exprimat arcum cuiuspiam lineae curvae respondentem abscissae vel cordae z, hinc plures arcus eiusdem curvae inter se comparare licet, ut vel differentia binorum arcuum fiat algebraica vel arcus exhibeantur datam rationem inter se tenentes. Hoc modo eiusmodi insignes curvarum proprietates eruuntur, quarum ratio aliunde vix perspici queat. Comparatio quidem arcuum circularium ex elementis nota per caput praecedens, ut vidimus, facile expeditur, unde etiam comparatio arcuum parabolicorum derivatur. Ex hoc autem capite comparatio arcuum ellipticorum et hyperbolicorum simili modo institui potest; cum enim in genere arcus sectionis conicae tali formula exprimatur

$$\int dx\sqrt{\frac{a+bxx}{c+exx}},$$

haec transformata in istam

$$\int\frac{dx(a+bxx)}{\sqrt{(ac+(ae+bc)xx+bex^4)}}$$

per praecepta tradita tractari potest ponendo $A=ac$, $C=ae+bc$, $E=be$ et $L=a$, $M=b$ atque $N=0$. Haec autem investigatio ad formulas, quarum denominator est $\sqrt{(A+2Bz+Czz+2Dz^3+Ez^4)}$, extendi potest similisque est praecedenti, quam idcirco hic sum expositurus, unde simul patebit hunc esse ultimum terminum, quousque progredi liceat. Formulae enim integrales magis complicatae, ubi post signum radicale altiores potestates ipsius z occurrunt vel ipsum signum radicale altiorem dignitatem involvit, hoc modo non videntur inter se comparari posse paucissimis casibus exceptis, qui per quampiam substitutionem ad huiusmodi formam reduci queant.

PROBLEMA 81

623. *Si $\Pi : z$ eiusmodi functionem ipsius z denotet, ut sit*

$$\Pi : z = \int\frac{dz}{\sqrt{(A+2Bz+Czz+2Dz^3+Ez^4)}},$$

huiusmodi functiones inter se comparare.

SOLUTIO

Inter binas variabiles x et y statuatur relatio hac aequatione expressa

$$\alpha+2\beta(x+y)+\gamma(xx+yy)+2\delta xy+2\varepsilon xy(x+y)+\zeta xxyy=0;$$

unde cum fiat

$$yy=\frac{-2y(\beta+\delta x+\varepsilon xx)-\alpha-2\beta x-\gamma xx}{\gamma+2\varepsilon x+\zeta xx},$$

erit radice extracta

$$y=\frac{-\beta-\delta x-\varepsilon xx+\sqrt{((\beta+\delta x+\varepsilon xx)^2-(\alpha+2\beta x+\gamma xx)(\gamma+2\varepsilon x+\zeta xx))}}{\gamma+2\varepsilon x+\zeta xx}.$$

Reducatur signum radicale ad formam propositam ponendo

$$\beta\beta-\alpha\gamma=Am,\qquad \beta\delta-\alpha\varepsilon-\beta\gamma=Bm,$$
$$\delta\delta-2\beta\varepsilon-\alpha\zeta-\gamma\gamma=Cm,\quad \delta\varepsilon-\beta\zeta-\gamma\varepsilon=Dm,$$
$$\varepsilon\varepsilon-\gamma\zeta=Em,$$

unde ex sex coefficientibus α, β, γ, δ, ε, ζ quinque definiuntur, atque ad sextum insuper accedit littera m, ita ut aequatio assumta adhuc constantem arbitrariam involvat. Inde ergo, si brevitatis gratia ponamus

$$\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}=X$$

et

$$\sqrt{(A+2By+Cyy+2Dy^3+Ey^4)}=Y,$$

habebimus

$$\beta+\gamma y+\delta x+\varepsilon xx+2\varepsilon xy+\zeta xxy=X\sqrt{m}$$

et

$$\beta+\gamma x+\delta y+\varepsilon yy+2\varepsilon xy+\zeta xyy=Y\sqrt{m}.$$

At aequatio assumta per differentiationem dat

$$dx(\beta+\gamma x+\delta y+2\varepsilon xy+\varepsilon yy+\zeta xyy)$$
$$+\,dy(\beta+\gamma y+\delta x+\varepsilon xx+2\varepsilon xy+\zeta xxy)=0;$$

quae expressiones quia cum superioribus conveniunt, dant

$$Ydx\sqrt{m}+Xdy\sqrt{m}=0 \quad \text{seu} \quad \frac{dx}{X}+\frac{dy}{Y}=0,$$

unde integrando colligimus

$$\Pi : x+\Pi : y=\text{Const.},$$

quae constans, si posito $x=0$ fiat $y=b$, erit $=\Pi:0+\Pi:b$, vel in genere si posito $x=a$ fiat $y=b$, ea erit $\Pi:a+\Pi:b$. Quodsi ergo litterae α, β, γ, δ, ε, ζ per conditiones superiores definiantur, aequatio assumta algebraica inter x et y erit integrale completum huius aequationis differentialis

$$\frac{dx}{\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}}+\frac{dy}{\sqrt{(A+2By+Cyy+2Dy^3+Ey^4)}}=0.$$

COROLLARIUM 1

624. Ad has litteras α, β, γ, δ, ε, ζ definiendas sumantur primo aequationes binae ad dextram positae, quae sunt

$$(\delta-\gamma)\beta-\alpha\varepsilon=Bm \quad \text{et} \quad (\delta-\gamma)\varepsilon-\zeta\beta=Dm,$$

unde quaerantur binae β et ε, reperieturque

$$\beta=\frac{(\delta-\gamma)B+\alpha D}{(\delta-\gamma)^2-\alpha\zeta}m \quad \text{et} \quad \varepsilon=\frac{(\delta-\gamma)D+\zeta B}{(\delta-\gamma)^2-\alpha\zeta}m.$$

COROLLARIUM 2

625. Sit brevitatis gratia $\delta - \gamma = \lambda$ seu $\delta = \gamma + \lambda$; erit

$$\beta = \frac{D\alpha + B\lambda}{\lambda\lambda - \alpha\xi} m \quad \text{et} \quad \varepsilon = \frac{B\xi + D\lambda}{\lambda\lambda - \alpha\xi} m.$$

Iam ex conditione prima et ultima oritur

$$\beta\beta\zeta - \alpha\varepsilon\varepsilon = (A\zeta - E\alpha)m,$$

ubi illi valores substituti praebent $\frac{BB\xi - DD\alpha}{\lambda\lambda - \alpha\xi} m = A\zeta - E\alpha$, unde fit

$$m = \frac{(\lambda\lambda - \alpha\xi)(A\xi - E\alpha)}{BB\xi - DD\alpha}.$$

At ex prima et ultima sequitur

$$DD\beta\beta - BB\varepsilon\varepsilon + \gamma(BB\zeta - DD\alpha) = (ADD - BBE)m,$$

unde colligitur

$$\gamma = \frac{(A\xi - E\alpha)((ADD - BBE)\lambda\lambda + 2BD(A\xi - E\alpha)\lambda + ABB\xi\xi - DDE\alpha\alpha)}{(BB\xi - DD\alpha)^2}.$$

COROLLARIUM 3

626. Superest tertia aequatio

$$2\gamma\lambda + \lambda\lambda - 2\beta\varepsilon - \alpha\zeta = Cm,$$

et cum pro m substituto valore sit

$$\beta = \frac{(A\xi - E\alpha)(D\alpha + B\lambda)}{BB\xi - DD\alpha} \quad \text{et} \quad \varepsilon = \frac{(A\xi - E\alpha)(B\xi + D\lambda)}{BB\xi - DD\alpha},$$

si isti valores substituantur, commode inde colligitur

$$\lambda = \frac{C(A\xi - E\alpha)(BB\xi - DD\alpha) - 2BD(A\xi - E\alpha)^2 - (BB\xi - DD\alpha)^2}{2(A\xi - E\alpha)(ADD - BBE)}.$$

SCHOLION 1

627. Quia his valoribus uti non licet, quoties fuerit $ADD - BBE = 0$, aliam resolutionem huic incommodo non obnoxiam tradam.

Posito $\delta = \gamma + \lambda$ sit insuper $\lambda\lambda = \alpha\zeta + \mu$, ut primae formulae fiant

$$\beta = \frac{m}{\mu}(D\alpha + B\lambda) \quad \text{et} \quad \varepsilon = \frac{m}{\mu}(B\zeta + D\lambda).$$

Iam prima et ultima iunctis prodit

$$A\zeta - E\alpha = \frac{m}{\mu}(BB\zeta - DD\alpha),$$

qua aequatione ratio inter α et ζ definitur; quae cum sufficiat, erit

$$\alpha = \mu A - BBm \quad \text{et} \quad \zeta = \mu E - DDm$$

hincque

$$\lambda\lambda = \mu + (\mu A - BBm)(\mu E - DDm),$$

unde colligimus

$$\gamma = \frac{mm}{\mu\mu}(2BD\lambda + (ADD + BBE)\mu) - \frac{2BBDDm^3}{\mu\mu} - \frac{m}{\mu}.$$

Valores α et ζ in formula Corollarii 3 substituti dant

$$\lambda = \frac{\mu\mu}{2m} + BDm - \frac{1}{2}C\mu,$$

cuius quadratum illi valori $\alpha\zeta + \mu$ aequatum perducit ad hanc aequationem

$$\mu(\mu - Cm)^2 + 4(BD - AE)mm\mu + 4(ADD - BCD + BBE)m^3 = 4mm;$$

ad quam resolvendam ponatur $\mu = Mm$ fietque

$$m = \frac{4}{M(M-C)^2 + 4M(BD-AE) + 4(ADD-BCD+BBE)}$$

atque hic est M constans illa arbitraria pro integrali completo requisita. Hoc modo omnes litterae α, β, γ, δ, ε, ζ eodem denominatore affecti prodibunt, quo omisso habebimus

$$\alpha = 4(AM - BB), \quad \beta = 2B(M - C) + 4AD, \quad \gamma = 4AE - (M - C)^2,$$
$$\zeta = 4(EM - DD), \quad \varepsilon = 2D(M - C) + 4BE, \quad \delta = MM - CC + 4(AE + BD),$$

quibus inventis aequatio nostra canonica

$$0 = \alpha + 2\beta(x + y) + \gamma(xx + yy) + 2\delta xy + 2\varepsilon xy(x + y) + \zeta xxyy,$$

si brevitatis gratia ponamus

$$M(M-C)^2+4M(BD-AE)+4(ADD-BCD+BBE)=\varDelta,$$

resoluta dabit

$$\beta+\delta x+\varepsilon xx+y(\gamma+2\varepsilon x+\zeta xx)=\pm 2\sqrt{\varDelta}(A+2Bx+Cxx+2Dx^3+Ex^4),$$
$$\beta+\delta y+\varepsilon yy+x(\gamma+2\varepsilon y+\zeta yy)=\pm 2\sqrt{\varDelta}(A+2By+Cyy+2Dy^3+Ey^4),$$

quae ergo est integrale completum huius aequationis differentialis

$$0=\frac{dx}{\pm\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}}+\frac{dy}{\pm\sqrt{(A+2By+Cyy+2Dy^3+Ey^4)}}.$$

SCHOLION 2

628. Cum hic ab idonea coefficientium determinatione totum negotium pendeat, operae pretium erit eam luculentius exponere. Posito igitur statim $\delta=\gamma+\lambda$ et $\lambda\lambda-\alpha\zeta=Mm$ quinque conditiones adimplendae sunt

$$\text{I. } \beta\beta-\alpha\gamma=Am,\quad \text{II. } \varepsilon\varepsilon-\gamma\zeta=Em,$$
$$\text{III. } \beta\lambda-\alpha\varepsilon=Bm,\quad \text{IV. } \varepsilon\lambda-\beta\zeta=Dm,$$
$$\text{V. } Mm+2\gamma\lambda-2\beta\varepsilon=Cm.$$

Hinc ex tertia et quarta combinando deducitur

$$m(B\lambda+D\alpha)=\beta(\lambda\lambda-\alpha\zeta)=\beta Mm,\quad \text{ergo}\quad \beta=\frac{B\lambda+D\alpha}{M},$$
$$m(D\lambda+B\zeta)=\varepsilon(\lambda\lambda-\alpha\zeta)=\varepsilon Mm,\quad \text{ergo}\quad \varepsilon=\frac{D\lambda+B\zeta}{M}.$$

Iam ex prima et secunda elidendo γ oritur

$$m(A\zeta-E\alpha)=\beta\beta\zeta-\varepsilon\varepsilon\alpha=\frac{BB\zeta-DD\alpha}{M}m$$

hincque

$$\zeta(AM-BB)=\alpha(EM-DD),$$

quare statuatur

$$\alpha=n(AM-BB)\quad \text{et}\quad \zeta=n(EM-DD).$$

Tum vero indidem est

$$E\beta\beta-E\alpha\gamma=A\varepsilon\varepsilon-A\gamma\zeta\quad \text{seu}\quad \gamma(A\zeta-E\alpha)=A\varepsilon\varepsilon-E\beta\beta;$$

pro qua tractanda cum sit pro α et ζ substitutis valoribus

$$\beta = nAD + \frac{B}{M}(\lambda - nBD) \quad \text{et} \quad \varepsilon = nBE + \frac{D}{M}(\lambda - nBD),$$

sit brevitatis ergo $\lambda - nBD = nMN$, ut habeamus

$$\beta = n(AD + BN) \quad \text{et} \quad \varepsilon = n(BE + DN),$$

et quia

$$A\zeta - E\alpha = n(BBE - ADD)$$

atque

$$A\varepsilon\varepsilon - E\beta\beta = nn(ABBEE + ADDNN - AADDE - BBENN)$$

seu

$$A\varepsilon\varepsilon - E\beta\beta = nn(BBE - ADD)(AE - NN),$$

fiet

$$\gamma = n(AE - NN).$$

Cum autem sit

$$\lambda = n(BD + MN) \quad \text{et} \quad \lambda\lambda = nn(AM - BB)(EM - DD) + Mm,$$

erit

$$Mm = nn(2BDMN + MMNN - AEMM + M(ADD + BBE))$$

seu

$$m = nn(2BDN + MNN - AEM + ADD + BBE).$$

Denique aequatio quinta $\beta\varepsilon - \gamma\lambda = \frac{1}{2}m(M - C)$ evoluta praebet

$$\begin{aligned}\beta\varepsilon - \gamma\lambda &= nn((AD + BN)(BE + DN) - (AE - NN)(BD + MN)) \\ &= nnN(2BDN + MNN - AEM + ADD + BBE) = Nm,\end{aligned}$$

unde fit $N = \frac{1}{2}(M - C)$, ac propterea

$$m = nn(BD(M - C) + \frac{1}{4}M(M - C)^2 - AEM + ADD + BBE).$$

Hincque sumendo $n = 4$ superiores valores obtinentur.

EXEMPLUM 1

629. *Invenire integrale completum huius aequationis differentialis*

$$\frac{dp}{\pm\sqrt{(a + bp)}} + \frac{dq}{\pm\sqrt{(a + bq)}} = 0.$$

Hic est $x=p$, $y=q$, $A=a$, $B=\frac{1}{2}b$, $C=0$, $D=0$, $E=0$, unde fiunt coefficientes

$$\alpha=4aM-bb,\quad \beta=bM,\quad \gamma=-MM,\quad \zeta=0,\quad \varepsilon=0,\quad \delta=MM$$

et

$$\varDelta=M^3,$$

unde integrale completum erit

$$bM+MMp-MMq=\pm 2M\sqrt{M(a+bp)}$$

seu

$$b+M(p-q)=\pm 2\sqrt{M(a+bp)}\quad \text{vel}\quad b+M(q-p)=\pm 2\sqrt{M(a+bq)},$$

quae signa ambigua radicalium cum signis in aequatione differentiali convenire debent.

EXEMPLUM 2

630. *Invenire integrale completum huius aequationis differentialis*

$$\frac{dp}{\pm\sqrt{(a+bp^2)}}+\frac{dq}{\pm\sqrt{(a+bq^2)}}=0.$$

Sumto $x=p$ et $y=q$ erit $A=a$, $B=0$, $C=b$, $D=0$, $[E=0]$, ergo

$$\alpha=4aM,\quad \beta=0,\quad \gamma=-(M-b)^2,\quad \zeta=0,\quad \varepsilon=0,\quad \delta=MM-bb$$

atque

$$\varDelta=M(M-b)^2,$$

unde integrale completum in his aequationibus continebitur

$$(MM-bb)p-(M-b)^2q=\pm 2(M-b)\sqrt{M(a+bpp)}$$

seu

$$(M+b)p-(M-b)q=\pm 2\sqrt{M(a+bpp)}$$

et

$$(M+b)q-(M-b)p=\pm 2\sqrt{M(a+bqq)}.$$

EXEMPLUM 3

631. *Invenire integrale completum huius aequationis differentialis*

$$\frac{dp}{\pm\sqrt{(a+bp^3)}}+\frac{dq}{\pm\sqrt{(a+bq^3)}}=0.$$

Sumto $x = p$, $y = q$ erit $A = a$, $B = 0$, $C = 0$, $D = \frac{1}{2}b$, $E = 0$, ergo

$$\alpha = 4aM, \quad \beta = 2ab, \quad \gamma = -MM, \quad \zeta = -bb, \quad \varepsilon = bM, \quad \delta = MM$$

et

$$\Delta = M^3 + abb,$$

unde integrale completum

$$2ab + MMp + bMpp + q(-MM + 2bMp - bbpp) = \pm 2\sqrt{(M^3 + abb)(a + bp^3)}$$

sive

$$2ab + Mp(M + bp) - q(M - bp)^2 = \pm 2\sqrt{(M^3 + abb)(a + bp^3)}$$

et

$$2ab + Mq(M + bq) - p(M - bq)^2 = \pm 2\sqrt{(M^3 + abb)(a + bq^3)}.$$

EXEMPLUM 4

632. *Invenire integrale completum huius aequationis differentialis*

$$\frac{dp}{\pm\sqrt{(a + bp^4)}} + \frac{dq}{\pm\sqrt{(a + bq^4)}} = 0.$$

Posito $x = p$, $y = q$ erit $A = a$, $B = 0$, $C = 0$, $D = 0$, $E = b$, ergo

$$\alpha = 4aM, \quad \beta = 0, \quad \gamma = 4ab - MM, \quad \zeta = 4bM, \quad \varepsilon = 0, \quad \delta = MM + 4ab$$

et

$$\Delta = M^3 - 4abM,$$

unde integrale completum

$$(MM + 4ab)p + q(4ab - MM + 4bMpp) = \pm 2\sqrt{M(MM - 4ab)(a + bp^4)},$$

$$(MM + 4ab)q + p(4ab - MM + 4bMqq) = \pm 2\sqrt{M(MM - 4ab)(a + bq^4)}.$$

EXEMPLUM 5

633. *Invenire integrale completum huius aequationis differentialis*

$$\frac{dp}{\pm\sqrt{(a + bp^6)}} + \frac{dq}{\pm\sqrt{(a + bq^6)}} = 0.$$

Ponatur $x = pp$ et $y = qq$ atque aequatio nostra generalis induet posito $A = 0$ hanc formam

$$\frac{dp}{\pm\sqrt{(2B + Cpp + 2Dp^4 + Ep^6)}} + \frac{dq}{\pm\sqrt{(2B + Cqq + 2Dq^4 + Eq^6)}} = 0.$$

Fieri ergo oportet $B=\frac{1}{2}a$, $C=0$, $D=0$ et $E=b$, unde coefficientes ita determinantur

$$\alpha=-aa,\quad \beta=aM,\quad \gamma=-MM,\quad \zeta=4bM,\quad \varepsilon=2ab,\quad \delta=MM$$

et

$$\Delta=M^3+aab,$$

ergo integrale completum

$$aM+MMpp+2abp^4+qq(-MM+4abpp+4bMp^4)$$
$$=\pm 2p\sqrt{(M^3+aab)(a+bp^6)}$$

sive

$$aM+MMqq+2abq^4+pp(-MM+4abqq+4bMq^4)$$
$$=\pm 2q\sqrt{(M^3+aab)(a+bq^6)}.$$

COROLLARIUM

634. Si sumatur constans $M=-\sqrt[3]{aab}$, ut sit $M^3+aab=0$, prodibit integrale particulare, quod ita se habebit

$$pp=\frac{qq\sqrt[3]{b}+\sqrt[3]{a}}{2qq\sqrt[3]{b}-\sqrt[3]{a}}\cdot\sqrt[3]{\frac{a}{b}}\quad \text{seu}\quad qq=\frac{pp\sqrt[3]{b}+\sqrt[3]{a}}{2pp\sqrt[3]{b}-\sqrt[3]{a}}\cdot\sqrt[3]{\frac{a}{b}},$$

quod aequationi differentiali utique satisfacit.

PROBLEMA 82

635. *Proposita hac aequatione differentiali*

$$\frac{dp}{\pm\sqrt{(a+bpp+cp^4+ep^6)}}+\frac{dq}{\pm\sqrt{(a+bqq+cq^4+eq^6)}}=0$$

eius integrale completum algebraice assignare.

SOLUTIO

Aequatio praecedens differentialis algebraice integrata ad hanc formam reducitur ponendo $x=pp$ et $y=qq$ atque $A=0$; prodibit enim

$$\frac{dp}{\pm\sqrt{(2B+Cpp+2Dp^4+Ep^6)}}+\frac{dq}{\pm\sqrt{(2B+Cqq+2Dq^4+Eq^6)}}=0.$$

Quare tantum opus est, ut fiat

$$A = 0, \quad B = \tfrac{1}{2}a, \quad C = b, \quad D = \tfrac{1}{2}c, \quad E = e,$$

unde coefficientes α, β, γ, δ, ε, ζ ita definientur

$$\alpha = -aa, \qquad \beta = a(M-b), \qquad \gamma = -(M-b)^2,$$
$$\zeta = 4eM - cc, \quad \varepsilon = c(M-b) + 2ae, \quad \delta = MM - bb + ac,$$
$$\varDelta = M(M-b)^2 + acM - abc + aae = (M-b)^3 + b(M-b)^2 + ac(M-b) + aae,$$

hincque integrale completum ob constantem M ab arbitrio nostro pendentem erit

$$\beta + \delta pp + \varepsilon p^4 + qq(\gamma + 2\varepsilon pp + \zeta p^4) = \pm 2p\sqrt{\varDelta(a + bpp + cp^4 + ep^6)},$$
$$\beta + \delta qq + \varepsilon q^4 + pp(\gamma + 2\varepsilon qq + \zeta q^4) = \pm 2q\sqrt{\varDelta(a + bqq + cq^4 + eq^6)},$$

quae binae quidem aequationes inter se conveniunt, sed ob ambiguitatem signorum in ipsa aequatione differentiali ambae notari debent ambiguitate inde sublata. Utrinque autem haec aequatio rationalis resultat

$$0 = \alpha + 2\beta(pp+qq) + \gamma(p^4+q^4) + 2\delta ppqq + 2\varepsilon ppqq(pp+qq) + \zeta p^4q^4.$$

COROLLARIUM 1

636. Si constans M ita sumatur, ut fiat $\varDelta = 0$, obtinetur integrale particulare huius formae

$$qq = \frac{E + Fpp}{G + Hpp},$$

quod etiam a posteriori cognoscere licet. Ut enim satisfaciat, sumi debet

$$aG^3 + bEGG + cEEG + eE^3 = 0,$$

unde ratio $E:G$ definitur; tum vero invenitur $F = -G$ et denique

$$H = \frac{-cEG - 2eEE}{aG} = \frac{2aGG + 2bEG + cEE}{aE}.$$

COROLLARIUM 2

637. Constans M ita mutetur, ut sit $M-b=\frac{a}{ff}$, fietque

$$\alpha=-aa, \qquad \beta=\frac{aa}{ff}, \qquad \gamma=-\frac{aa}{f^4},$$

$$\zeta=4be-cc+\frac{4ae}{ff}, \quad \varepsilon=\frac{ac}{ff}+2ae, \quad \delta=\frac{aa}{f^4}+\frac{2ab}{ff}+ac$$

et

$$\varDelta=\frac{aa}{f^6}(a+bff+cf^4+ef^6)$$

et aequatio integralis erit

$$\begin{aligned} &aaff+a(a+2bff+cf^4)pp+aff(c+2eff)p^4\\ &-qq(aa-2aff(c+2eff)pp+ff(ccff-4beff-4ae)p^4)\\ &=\pm 2afp\sqrt{(a+bff+cf^4+ef^6)(a+bpp+cp^4+ep^6)},\end{aligned}$$

unde patet posito $p=0$ fore $qq=ff$.

COROLLARIUM 3

638. Haec aequatio facile in hanc formam transmutatur

$$\begin{aligned} &aff(a+bpp+cp^4+ep^6)+app(a+bff+cf^4+ef^6)\\ &-qq(a-cffpp)^2-aeffpp(ff-pp)^2+4effppqq(aff+app+bffpp)\\ &=\pm 2fp\sqrt{a(a+bff+cf^4+ef^6)a(a+bpp+cp^4+ep^6)},\end{aligned}$$

unde statim patet, si sit $e=0$, fore hanc aequationem radicem extrahendo

$$f\sqrt{a(a+bpp+cp^4)}\mp p\sqrt{a(a+bff+cf^4)}=q(a-cffpp),$$

quae est integralis completa huius differentialis

$$\frac{dp}{\pm\sqrt{(a+bpp+cp^4)}}+\frac{dq}{\pm\sqrt{(a+bqq+cq^4)}}=0,$$

prorsus ut supra [§ 607] iam invenimus.

COROLLARIUM 4

639. Simili modo patet in genere, quando e non evanescit, integrale completum ita commodius exprimi posse

$$(f\sqrt{a(a+bpp+cp^4+ep^6)} \mp p\sqrt{a(a+bff+cf^4+ef^6)})^2$$
$$= qq(a-cffpp)^2 + aeffpp(ff-pp)^2 - 4effppqq(aff+app+bffpp),$$

quae ergo, cum posito $p=0$ fiat $q=f$, respondet huic functionum transcendentium relationi

$$\pm \Pi : p \pm \Pi : q = \pm \Pi : 0 \pm \Pi : f.$$

SCHOLION 1

640. Genera igitur functionum transcendentium, quas hoc modo perinde atque arcus circulares inter se comparare licet, in his binis formulis integralibus continentur

$$\int \frac{dz}{\sqrt{(A+2Bz+Czz+2Dz^3+Ez^4)}} \quad \text{et} \quad \int \frac{dz}{\sqrt{(a+bzz+cz^4+ez^6)}}$$

neque haec methodus ad alias formas magis complexas extendi posse videtur. Neque etiam posterior in denominatore potestates impares ipsius z admittit, nisi forte simplex substitutio reductioni ad illam formam sufficiat. Facile autem patet huiusmodi formam

$$\int \frac{dz}{\sqrt{(A+2Bz+Czz+2Dz^3+Ez^4+2Fz^5+Gz^6)}}$$

hac methodo tractari certe non posse. Si enim coefficientes ita essent comparati, ut radicis extractio succederet, talis formula

$$\int \frac{dz}{a+bz+czz+ez^3}$$

prodiret; cuius integratio cum tam logarithmos quam arcus circulares involvat, fieri omnino nequit, ut plures huiusmodi functiones algebraice inter se comparentur. Caeterum prior formula latius patet quam posterior, cum haec ex illa nascatur posito $A=0$, si zz loco z scribatur. De priori autem notari meretur, quod eandem formam servet, etiamsi transformetur hac substitutione

$$z=\frac{\alpha+\beta y}{\gamma+\delta y};$$

prodit enim

$$\int\frac{(\beta\gamma-\alpha\delta)dy}{\sqrt{(A(\gamma+\delta y)^4+2B(\alpha+\beta y)(\gamma+\delta y)^3+C(\alpha+\beta y)^2(\gamma+\delta y)^2+2D(\alpha+\beta y)^3(\gamma+\delta y)+E(\alpha+\beta y)^4)}},$$

ex quo intelligitur quantitates α, β, γ, δ ita accipi posse, ut potestates impares evanescant. Vel etiam ita definiri poterunt, ut terminus primus et ultimus evanescat; tum enim posito $y=uu$ iterum forma a potestatibus imparibus immunis nascitur.

SCHOLION 2

641. Sublatio autem potestatum imparium ita commodissime instituitur. Cum formula

$$A+2Bz+Czz+2Dz^3+Ez^4$$

certe semper habeat duos factores reales, ita exhibeatur formula integralis

$$\int\frac{dz}{\sqrt{(a+2bz+czz)(f+2gz+hzz)}},$$

quae posito $z=\frac{\alpha+\beta y}{\gamma+\delta y}$ abit in

$$\int\frac{(\beta\gamma-\alpha\delta)dy}{\sqrt{(a(\gamma+\delta y)^2+2b(\alpha+\beta y)(\gamma+\delta y)+c(\alpha+\beta y)^2)(f(\gamma+\delta y)^2+2g(\alpha+\beta y)(\gamma+\delta y)+h(\alpha+\beta y)^2)}},$$

ubi denominatoris factores evoluti sunt

$$(a\gamma\gamma+2b\alpha\gamma+c\alpha\alpha)+2(a\gamma\delta+b\alpha\delta+b\beta\gamma+c\alpha\beta)y+(a\delta\delta+2b\beta\delta+c\beta\beta)yy,$$
$$(f\gamma\gamma+2g\alpha\gamma+h\alpha\alpha)+2(f\gamma\delta+g\alpha\delta+g\beta\gamma+h\alpha\beta)y+(f\delta\delta+2g\beta\delta+h\beta\beta)yy;$$

quodsi iam utroque terminus medius evanescens reddatur, fit

$$\frac{\delta}{\beta}=\frac{-b\gamma-c\alpha}{a\gamma+b\alpha}=\frac{-g\gamma-h\alpha}{f\gamma+g\alpha}$$

hincque

$$bf\gamma\gamma+(bg+cf)\alpha\gamma+cg\alpha\alpha=ag\gamma\gamma+(ah+bg)\alpha\gamma+bh\alpha\alpha$$

seu

$$\gamma\gamma=\frac{(ah-cf)\alpha\gamma+(bh-cg)\alpha\alpha}{bf-ag},$$

unde fit

$$\frac{\gamma}{\alpha} = \frac{ah - cf + \sqrt{((ah-cf)^2 + 4(bf-ag)(bh-cg))}}{2(bf-ag)}.$$

Hinc sufficere posset eas tantum formulas, in quibus potestates impares desunt, tractasse, id quod initio huius capitis fecimus, sed si insuper numerator accedat, haec reductio non amplius locum habet.

PROBLEMA 83

642. *Denotante n numerum integrum quemcunque invenire integrale completum algebraice expressum huius aequationis differentialis*

$$\frac{dy}{\sqrt{(A + 2By + Cyy + 2Dy^3 + Ey^4)}} = \frac{ndx}{\sqrt{(A + 2Bx + Cxx + 2Dx^3 + Ex^4)}}.$$

SOLUTIO

Per functiones transcendentes integrale completum est

$$\Pi : y = n\Pi : x + \text{Const.}$$

At ut idem algebraice expressum eruamus, posito $M - C = L$ sit per formulas supra (§ 627) inventas

$$\alpha = 4(AC - BB + AL), \quad \beta = 4AD + 2BL, \quad \gamma = 4AE - LL,$$
$$\zeta = 4(CE - DD + EL), \quad \varepsilon = 4BE + 2DL, \quad \delta = 4AE + 4BD + 2CL + LL$$

et

$$\Delta = L^3 + CLL + 4(BD - AE)L + 4(ADD + BBE - ACE).$$

Quibus positis si fuerit

$$\beta + \delta p + \varepsilon pp + q(\gamma + 2\varepsilon p + \zeta pp) = \quad 2\sqrt{\Delta(A + 2Bp + Cpp + 2Dp^3 + Ep^4)},$$
$$\beta + \delta q + \varepsilon qq + p(\gamma + 2\varepsilon q + \zeta qq) = -2\sqrt{\Delta(A + 2Bq + Cqq + 2Dq^3 + Eq^4)},$$

erit $\Pi : q = \Pi : p + \text{Const.}$

Cum autem hae duae aequationes inter se conveniant et in hac rationali contineantur

$$\alpha+2\beta(p+q)+\gamma(pp+qq)+2\delta pq+2\varepsilon pq(p+q)+\zeta ppqq=0,$$

si sumamus posito $p=a$ fieri $q=b$, constans illa L ita definiri debet, ut sit

$$\alpha+2\beta(a+b)+\gamma(aa+bb)+2\delta ab+2\varepsilon ab(a+b)+\zeta aabb=0,$$

eritque

$$\Pi:q=\Pi:p+\Pi:b-\Pi:a,$$

ubi iam nullum inest discrimen inter constantes et variabiles. Ponamus ergo $b=p$, ut sit

$$\Pi:q=2\Pi:p-\Pi:a,$$

atque huic aequationi superiores aequationes algebraicae conveniunt, si modo quantitas L ita definiatur, ut sit

$$\alpha+2\beta(a+p)+\gamma(aa+pp)+2\delta ap+2\varepsilon ap(a+p)+\zeta aapp=0,$$

unde deducitur

$$\frac{1}{2}L(p-a)^2=A+B(a+p)+Cap+Dap(a+p)+Eaapp$$
$$\pm\sqrt{(A+2Ba+Caa+2Da^3+Ea^4)(A+2Bp+Cpp+2Dp^3+Ep^4)}.$$

Hoc ergo valore pro L constituto indeque litteris α, β, γ, δ, ε, ζ per superiores formulas rite definitis, si iam p et q ut variabiles, a vero ut constantem spectemus, erit haec aequatio

$$\alpha+2\beta(p+q)+\gamma(pp+qq)+2\delta pq+2\varepsilon pq(p+q)+\zeta ppqq=0$$

integrale completum huius aequationis differentialis

$$\frac{dq}{\sqrt{(A+2Bq+Cqq+2Dq^3+Eq^4)}}=\frac{2dp}{\sqrt{(A+2Bp+Cpp+2Dp^3+Ep^4)}}.$$

Postquam hoc modo q per p definivimus, determinetur r per hanc aequationem

$$\alpha+2\beta(q+r)+\gamma(qq+rr)+2\delta qr+2\varepsilon qr(q+r)+\zeta qqrr=0;$$

erit

$$\Pi:r-\Pi:q=\Pi:p-\Pi:a,$$

quoniam posito $q = a$ et $r = p$ littera L, quae in valores α, β, γ, δ, ε, ζ ingreditur, perinde definitur ut ante. Quare cum sit $\Pi : q = 2\Pi : p - \Pi : a$, erit

$$\Pi : r = 3\Pi : p - 2\Pi : a,$$

unde sumto a constante illa aequatio algebraica inter q et r, dum q per praecedentem aequationem ex p definitur, erit integrale completum huius aequationis differentialis

$$\frac{dr}{\sqrt{(A + 2Br + Crr + 2Dr^3 + Er^4)}} = \frac{3dp}{\sqrt{(A + 2Bp + Cpp + 2Dp^3 + Ep^4)}}.$$

Hoc valore ipsius r per p invento quaeratur s per hanc aequationem

$$\alpha + 2\beta(r + s) + \gamma(rr + ss) + 2\delta rs + 2\varepsilon rs(r + s) + \zeta rrss = 0$$

retinente L semper valorem primo assignatum eritque

$$\Pi : s - \Pi : r = \Pi : p - \Pi : a \quad \text{seu} \quad \Pi : s = 4\Pi : p - 3\Pi : a,$$

unde ista aequatio algebraica erit integrale completum huius aequationis differentialis

$$\frac{ds}{\sqrt{(A + 2Bs + Css + 2Ds^3 + Es^4)}} = \frac{4dp}{\sqrt{(A + 2Bp + Cpp + 2Dp^3 + Ep^4)}}.$$

Cum hoc modo, quousque libuerit, progredi liceat, perspicuum est ad integrale completum huius aequationis differentialis inveniendum

$$\frac{dz}{\sqrt{(A + 2Bz + Czz + 2Dz^3 + Ez^4)}} = \frac{ndp}{\sqrt{(A + 2Bp + Cpp + 2Dp^3 + Ep^4)}}$$

sequentes operationes institui oportere:

1) Quaeratur quantitas L, ut sit

$$\frac{1}{2}L(p - a)^2 = A + B(a + p) + Cap + Dap(a + p) + Eaapp$$
$$\pm\sqrt{(A + 2Ba + Caa + 2Da^3 + Ea^4)(A + 2Bp + Cpp + 2Dp^3 + Ep^4)}.$$

2) Hinc determinentur litterae α, β, γ, δ, ε, ζ per has formulas

$$\alpha = 4(AC - BB + AL), \quad \beta = 4AD + 2BL, \quad \gamma = 4AE - LL,$$
$$\zeta = 4(CE - DD + EL), \quad \varepsilon = 4BE + 2DL, \quad \delta = 4AE + 4BD + 2CL + LL.$$

3) Formetur series quantitatum $p, q, r, s, t, \ldots z$, quarum prima sit p, secunda q, tertia r etc., ultima vero ordine n sit z, quae successive per has aequationes determinentur

$$\alpha + 2\beta(p+q) + \gamma(pp+qq) + 2\delta pq + 2\varepsilon pq(p+q) + \zeta ppqq = 0,$$
$$\alpha + 2\beta(q+r) + \gamma(qq+rr) + 2\delta qr + 2\varepsilon qr(q+r) + \zeta qqrr = 0,$$
$$\alpha + 2\beta(r+s) + \gamma(rr+ss) + 2\delta rs + 2\varepsilon rs(r+s) + \zeta rrss = 0$$

etc.,

donec ad ultimam z perveniatur.

4) Relatio, quae hinc concluditur inter p et z, erit integrale completum aequationis differentialis propositae et littera a vicem gerit constantis arbitrariae per integrationem ingressae.

COROLLARIUM

643. Hinc etiam integrale completum inveniri potest huius aequationis differentialis

$$\frac{m\,dy}{\sqrt{(A+2By+Cyy+2Dy^3+Ey^4)}} = \frac{n\,dx}{\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}}$$

designantibus m et n numeros integros. Statuatur enim utrumque membrum

$$= \frac{du}{\sqrt{(A+2Bu+Cuu+2Du^3+Eu^4)}}$$

et quaeratur relatio tam inter x et u quam inter y et u; unde elisa u orietur aequatio algebraica inter x et y.

SCHOLION

644. Ne hic extractio radicis in singulis aequationibus repetenda ambiguitatem creet, loco uniuscuiusque uti conveniet binis per extractionem iam erutis. Scilicet ut ex prima valor q rite per p definiatur, primo quidem habemus

$$q = \frac{-\beta - \delta p - \varepsilon pp + 2\sqrt{\Delta(A+2Bp+Cpp+2Dp^3+Ep^4)}}{\gamma + 2\varepsilon p + \zeta pp},$$

tum vero capi debet

$$2\sqrt{\Delta(A+2Bq+Cqq+2Dq^3+Eq^4)} = -\beta - \delta q - \varepsilon qq - p(\gamma + 2\varepsilon q + \zeta qq)$$

similique modo in relatione inter binas sequentes quantitates investiganda erit procedendum.

Caeterum adhuc notari convenit numeros integros m et n positivos esse debere neque hanc investigationem ad negativos extendi, propterea quod formula differentialis

$$\frac{dz}{\sqrt{(A+2Bz+Czz+2Dz^3+Ez^4)}}$$

posito z negativo naturam suam mutat. Interim tamen cum hanc aequalitatem $\Pi:x+\Pi:y=$ Const. supra algebraice expresserimus, eius ope quoque ii casus resolvi possunt, ubi est m vel n numerus negativus; si enim fuerit $\Pi:z=n\Pi:p+C$, quaeratur y, ut sit $\Pi:y+\Pi:z=$ Const., eritque

$$\Pi:y=-n\Pi:p+\text{Const.}$$

PROBLEMA 84

645. *Si $\Pi:z$ eiusmodi functionem transcendentem ipsius z denotet, ut sit*

$$\Pi:z=\int\frac{dz(\mathfrak{A}+\mathfrak{B}z+\mathfrak{C}zz+\mathfrak{D}z^3+\mathfrak{E}z^4)}{\sqrt{(A+2Bz+Czz+2Dz^3+Ez^4)}},$$

comparationem inter huiusmodi functiones investigare.

SOLUTIO

Ex coefficientibus A, B, C, D, E una cum constante arbitraria L determinentur sequentes valores

$$\alpha=4(AC-BB+AL),\quad \beta=4AD+2BL,\quad \gamma=4AE-LL,$$
$$\zeta=4(CE-DD+EL),\quad \varepsilon=4BE+2DL,\quad \delta=4AE+4BD+2CL+LL$$

et inter binas variabiles x et y haec constituatur relatio

$$\alpha+2\beta(x+y)+\gamma(xx+yy)+2\delta xy+2\varepsilon xy(x+y)+\zeta xxyy=0$$

eritque

$$\frac{dx}{\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}}+\frac{dy}{\sqrt{(A+2By+Cyy+2Dy^3+Ey^4)}}=0,$$

pro qua sine ambiguitate habetur

$$\beta+\delta x+\varepsilon xx+y(\gamma+2\varepsilon x+\zeta xx)=2\sqrt{\varDelta}(A+2Bx+Cxx+2Dx^3+Ex^4),$$

$$\beta+\delta y+\varepsilon yy+x(\gamma+2\varepsilon y+\zeta yy)=2\sqrt{\varDelta}(A+2By+Cyy+2Dy^3+Ey^4)$$

existente

$$\varDelta=L^3+CL^2+4(BD-AE)L+4(ADD+BBE-ACE).$$

Quare si ponamus

$$\frac{dx(\mathfrak{A}+\mathfrak{B}x+\mathfrak{C}x^2+\mathfrak{D}x^3+\mathfrak{E}x^4)}{\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}}+\frac{dy(\mathfrak{A}+\mathfrak{B}y+\mathfrak{C}y^2+\mathfrak{D}y^3+\mathfrak{E}y^4)}{\sqrt{(A+2By+Cyy+2Dy^3+Ey^4)}}=2dV\sqrt{\varDelta},$$

ut sit

$$\varPi:x+\varPi:y=\text{Const.}+2V\sqrt{\varDelta},$$

erit

$$\frac{dx(\mathfrak{B}(x-y)+\mathfrak{C}(x^2-y^2)+\mathfrak{D}(x^3-y^3)+\mathfrak{E}(x^4-y^4))}{\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}}=2dV\sqrt{\varDelta}$$

seu

$$dV=\frac{dx(\mathfrak{B}(x-y)+\mathfrak{C}(x^2-y^2)+\mathfrak{D}(x^3-y^3)+\mathfrak{E}(x^4-y^4))}{\beta+\delta x+\varepsilon xx+y(\gamma+2\varepsilon x+\zeta xx)}.$$

Ponatur nunc $x+y=t$ et $xy=u$, et quia $dx+dy=dt$ et $xdy+ydx=du$, erit $dx=\frac{xdt-du}{x-y}$ seu $(x-y)dx=xdt-du$, tum vero est $x=\frac{1}{2}t+\sqrt{\left(\frac{1}{4}tt-u\right)}$. At his positionibus aequatio assumta induit hanc formam

$$\alpha+2\beta t+\gamma tt+2(\delta-\gamma)u+2\varepsilon tu+\zeta uu=0,$$

unde fit differentiando

$$dt(\beta+\gamma t+\varepsilon u)+du(\delta-\gamma+\varepsilon t+\zeta u)=0,$$

ergo $dt=\frac{-du(\delta-\gamma+\varepsilon t+\zeta u)}{\beta+\gamma t+\varepsilon u}$ et

$$xdt-du=\frac{-du(\beta+\gamma t+\varepsilon u+(\delta-\gamma)x+\varepsilon tx+\zeta ux)}{\beta+\gamma t+\varepsilon u}$$

sive

$$xdt-du=\frac{-du(\beta+\delta x+\varepsilon xx+y(\gamma+2\varepsilon x+\zeta xx))}{\beta+\gamma t+\varepsilon u},$$

sicque habebimus

$$\frac{dx(x-y)}{\beta+\delta x+\varepsilon xx+y(\gamma+2\varepsilon x+\zeta xx)}=\frac{-du}{\beta+\gamma t+\varepsilon u},$$

ergo

$$dV = \frac{-du(\mathfrak{B} + \mathfrak{C}t + \mathfrak{D}(tt-u) + \mathfrak{E}t(tt-2u))}{\beta + \gamma t + \varepsilon u}$$

seu

$$dV = \frac{+dt(\mathfrak{B} + \mathfrak{C}t + \mathfrak{D}(tt-u) + \mathfrak{E}t(tt-2u))}{\delta - \gamma + \varepsilon t + \zeta u}.$$

Est vero aequatione illa resoluta

$$t = \frac{-\beta - \varepsilon u + \sqrt{(\beta\beta - \alpha\gamma + 2(\gamma\gamma + \beta\varepsilon - \gamma\delta)u + (\varepsilon\varepsilon - \gamma\zeta)uu)}}{\gamma}$$

seu [§ 628]

$$t = \frac{-\beta - \varepsilon u + 2\sqrt{\varDelta(A + Lu + Euu)}}{\gamma},$$

unde conficitur

$$dV = \frac{-du(\mathfrak{B} + \mathfrak{C}t + \mathfrak{D}(tt-u) + \mathfrak{E}t(tt-2u))}{2\sqrt{\varDelta(A + Lu + Euu)}}$$

ideoque

$$\Pi : x + \Pi : y = \text{Const.} - \int \frac{du(\mathfrak{B} + \mathfrak{C}t + \mathfrak{D}(tt-u) + \mathfrak{E}t(tt-2u))}{\sqrt{(A + Lu + Euu)}}.$$

Vel cum reperiatur

$$u = \frac{-(\delta-\gamma) - \varepsilon t + \sqrt{((\delta-\gamma)^2 - \alpha\zeta + 2((\delta-\gamma)\varepsilon - \beta\zeta)t + (\varepsilon\varepsilon - \gamma\zeta)tt)}}{\zeta},$$

quae expressio abit in hanc [§ 628]

$$u = \frac{-(\delta-\gamma) - \varepsilon t + 2\sqrt{\varDelta(L + C + 2Dt + Ett)}}{\zeta},$$

inde fit

$$dV = \frac{dt(\mathfrak{B} + \mathfrak{C}t + \mathfrak{D}(tt-u) + \mathfrak{E}t(tt-2u))}{2\sqrt{\varDelta(L + C + 2Dt + Ett)}}$$

sicque habebimus per t

$$\Pi : x + \Pi : y = \text{Const.} + \int \frac{dt(\mathfrak{B} + \mathfrak{C}t + \mathfrak{D}(tt-u) + \mathfrak{E}t(tt-2u))}{\sqrt{(L + C + 2Dt + Ett)}},$$

quae expressio, nisi sit algebraica, certe vel per logarithmos vel arcus circulares exhiberi potest. Tum vero post integrationem tantum opus est, ut loco t restituatur eius valor $x + y$.

COROLLARIUM 1

646. Si velimus, ut posito $x = a$ fiat $y = b$, constans L ita debet definiri, ut sit

$$\tfrac{1}{2}L(a-b)^2 = A + B(a+b) + Cab + Dab(a+b) + Eaabb$$
$$\pm\sqrt{(A + 2Ba + Caa + 2Da^3 + Ea^4)(A + 2Bb + Cbb + 2Db^3 + Eb^4)};$$

tum igitur constans nostra erit $= \Pi : a + \Pi : b$ integrali postremo ita sumto, ut evanescat posito $t = a + b$.

COROLLARIUM 2

647. Eodem modo etiam differentia functionum $\Pi : x - \Pi : y$ exprimi potest mutando alterutrius formulae radicalis signum, quo pacto formularum differentialium signum alterius convertetur.

COROLLARIUM 3

648. Quantitas V comparationi harum functionum inserviens erit algebraica, si haec formula differentialis

$$\frac{dt(\mathfrak{B}\zeta + \mathfrak{C}\zeta t + \mathfrak{D}(\delta - \gamma + \varepsilon t + \zeta tt) + \mathfrak{E}(2(\delta-\gamma) + 2\varepsilon t + \zeta tt)t)}{\zeta\sqrt{(L + C + 2Dt + Ett)}}$$

integrationem admittat, quia altera pars $\frac{-2dt\sqrt{\Delta}}{\zeta}(\mathfrak{D} + 2\mathfrak{E}t)$ per se est integrabilis.

SCHOLION

649. Hoc ergo argumentum plane novum de comparatione huiusmodi functionum transcendentium tam copiose pertractavimus, quam praesens institutum postulare videbatur. Quando autem eius applicatio ad comparationem arcuum curvarum, quorum longitudo huiusmodi functionibus exprimitur, erit facienda, uberiori evolutione erit opus, ubi contemplatio singularium proprietatum, quae hoc modo eruuntur, eximium usum afferre poterit. Commode autem hoc argumentum ad doctrinam de resolutione aequationum differentialium referri videtur, siquidem inde eiusmodi aequationum integralia completa et quidem algebraice exhiberi possunt, quae aliis methodis frustra indagantur. Nunc igitur huic sectioni finem faciet methodus generalis omnium aequationum differentialium integralia proxime determinandi.

CAPUT VII

DE INTEGRATIONE AEQUATIONUM DIFFERENTIALIUM PER APPROXIMATIONEM

PROBLEMA 85

650. *Proposita aequatione differentiali quacunque eius integrale completum vero proxime assignare.*

SOLUTIO

Sint x et y binae variabiles, inter quas aequatio differentialis proponitur, atque haec aequatio huiusmodi habebit formam, ut sit $\frac{dy}{dx} = V$ existente V functione quacunque ipsarum x et y. Iam cum integrale completum desideretur, hoc ita est interpretandum, ut, dum ipsi x certus quidem valor, puta $x = a$, tribuitur, altera variabilis y datum quemdam valorem, puta $y = b$, adipiscatur. Quaestionem ergo primo ita tractemus, ut investigemus valorem ipsius y, quando ipsi x valor paulisper ab a discrepans tribuitur, seu posito $x = a + \omega$ ut quaeramus y. Cum autem ω sit particula minima, etiam valor ipsius y minime a b discrepabit; unde, dum x ab a usque ad $a + \omega$ tantum mutatur, quantitatem V interea tanquam constantem spectare licet. Quare posito $x = a$ et $y = b$ fiat $V = A$ et pro hac exigua mutatione habebimus $\frac{dy}{dx} = A$ ideoque integrando $y = b + A(x - a)$, eiusmodi scilicet constante adiecta, ut posito $x = a$ fiat $y = b$. Statuamus ergo $x = a + \omega$ fietque $y = b + A\omega$.

Quemadmodum ergo hic ex valoribus initio datis $x = a$ et $y = b$ proxime sequentes $x = a + \omega$ et $y = b + A\omega$ invenimus, ita ab his simili modo per intervalla minima ulterius progredi licet, quoad tandem ad valores a primitivis quantumvis remotos perveniatur. Quae operationes quo clarius ob oculos ponantur, sequenti modo successive instituantur.

Ipsius	valores successivi
x	$a,\ a',\ a'',\ a''',\ a^{\mathrm{IV}},\ \ldots\ 'x,\ x$
y	$b,\ b',\ b'',\ b''',\ b^{\mathrm{IV}},\ \ldots\ 'y,\ y$
V	$A,\ A',\ A'',\ A''',\ A^{\mathrm{IV}},\ \ldots\ 'V,\ V.$

Scilicet ex primis $x = a$ et $y = b$ datis habetur $V = A$, tum vero pro secundis erit $b' = b + A(a' - a)$ differentia $a' - a$ minima pro lubitu assumta. Hinc ponendo $x = a'$ et $y = b'$ colligitur $V = A'$ indeque pro tertiis obtinebitur $b'' = b' + A'(a'' - a')$, ubi posito $x = a''$ et $y = b''$ invenitur $V = A''$. Iam pro quartis habebimus $b''' = b'' + A''(a''' - a'')$ hincque ponendo $x = a'''$ et $y = b'''$ colligemus $V = A'''$ sicque ad valores a primitivis quantumvis remotos progredi licebit. Series autem prima valores ipsius x successivos exhibens pro lubitu accipi potest, dummodo per intervalla minima ascendat vel etiam descendat.

COROLLARIUM 1

651. Pro singulis ergo intervallis minimis calculus eodem modo instituitur sicque valores, a quibus sequentia pendent, obtinentur. Hoc ergo modo singulis pro x assumtis valoribus valores respondentes ipsius y assignari possunt.

COROLLARIUM 2

652. Quo minora accipiuntur intervalla, per quae valores ipsius x progredi assumuntur, eo accuratius valores pro singulis eliciuntur. Interim tamen errores in singulis commissi, etiamsi sint multo minores, ob multitudinem coacervantur.

COROLLARIUM 3

653. Errores autem in hoc calculo inde oriuntur, quod in singulis intervallis ambas quantitates x et y ut constantes spectemus sicque functio V pro constante habeatur. Quo magis ergo valor ipsius V a quovis intervallo ad sequens immutatur, eo maiores errores sunt pertimescendi.

SCHOLION 1

654. Hoc incommodum imprimis occurrit, ubi valor ipsius V vel evanescit vel in infinitum excrescit, etiamsi mutationes ipsis x et y accidentes sint satis parvae. His autem casibus errores saltim enormes sequenti modo evitabuntur: Sit pro initio huiusmodi intervalli $x = a$ et $y = b$, tum vero in ipsa aequatione proposita ponatur $x = a + \omega$ et $y = b + \psi$, ut sit $\frac{d\psi}{d\omega} = V$, in V autem ita fiat substitutio $x = a + \omega$ et $y = b + \psi$, ut quantitates ω et ψ tanquam minimae spectentur, reiiciendo scilicet altiores potestates prae inferioribus; hoc enim modo plerumque integratio pro his intervallis actu institui poterit. Hac autem emendatione vix unquam erit opus, nisi termini ex ipsis valoribus a et b nati se destruant. Veluti si habeatur haec aequatio

$$\frac{dy}{dx} = \frac{aa}{xx - yy}$$

ac pro initio debeat esse $x = a$ et $y = a$; iam pro intervallo hinc incipiente ponatur $x = a + \omega$ et $y = a + \psi$ habebiturque

$$\frac{d\psi}{d\omega} = \frac{aa}{2a\omega - 2a\psi}$$

seu $2\omega d\psi - 2\psi d\psi = a d\omega$ seu $d\omega - \frac{2\omega d\psi}{a} = \frac{-2\psi d\psi}{a}$, quae per $e^{\frac{-2\psi}{a}} = 1 - \frac{2\psi}{a}$ multiplicata et integrata praebet

$$\left(1 - \frac{2\psi}{a}\right)\omega = \frac{-2}{a}\int\left(1 - \frac{2\psi}{a}\right)\psi d\psi = -\frac{\psi\psi}{a},$$

quia posito $\omega = 0$ fieri debet $\psi = 0$. Hinc ergo habetur $\omega = \frac{-\psi\psi}{a - 2\psi} = \frac{-\psi\psi}{a}$ seu $a(a' - a) = -(b' - b)^2$ existente $b = a$, unde colligitur pro sequente intervallo $b' = b + \sqrt{-a(a' - a)}$, quo casu patet valorem x non ultra a augeri posse, quia y fieret imaginarium.

SCHOLION 2

655. Passim traduntur regulae aequationum differentialium integralia per series infinitas exprimendi, quae autem plerumque hoc vitio laborant, ut integralia tantum particularia exhibeant, praeterquam quod series illae certo tantum casu convergant neque ergo aliis casibus ullum usum praestent. Veluti si proposita sit aequatio

$$dy + y dx = a x^n dx,$$

iubemur huiusmodi seriem in genere fingere

$$y = Ax^{\alpha} + Bx^{\alpha+1} + Cx^{\alpha+2} + Dx^{\alpha+3} + Ex^{\alpha+4} + \text{etc.},$$

qua substituta fit

$$\left.\begin{array}{l} \alpha A x^{\alpha-1} + (\alpha+1)Bx^{\alpha} + (\alpha+2)Cx^{\alpha+1} + (\alpha+3)Dx^{\alpha+2} + \text{etc.} \\ \qquad\quad + \qquad\quad A \quad + \qquad B \quad + \qquad C \\ - ax^{n} \end{array}\right\} = 0.$$

Statuatur ergo $\alpha - 1 = n$ seu $\alpha = n + 1$ eritque $A = \frac{a}{n+1}$, tum vero reliquis terminis ad nihilum reductis $B = \frac{-A}{n+2}$, $C = \frac{-B}{n+3}$, $D = \frac{-C}{n+4}$ etc. sicque habebitur haec series

$$y = \frac{ax^{n+1}}{n+1} - \frac{ax^{n+2}}{(n+1)(n+2)} + \frac{ax^{n+3}}{(n+1)(n+2)(n+3)} - \frac{ax^{n+4}}{(n+1)(n+2)(n+3)(n+4)} + \text{etc.}$$

Verum hoc integrale tantum est particulare, quoniam evanescente x simul y evanescit, nisi $n + 1$ sit numerus negativus; tum vero haec series non convergit, nisi x capiatur valde parvum. Quamobrem hinc minime cognoscere licet valores ipsius y, qui respondeant valoribus quibuscunque ipsius x. Hoc autem vitio non laborat methodus, quam hic adumbravimus, cum primo integrale completum praebeat, dum scilicet pro dato ipsius x valore datum ipsi y valorem tribuit, tum vero per intervalla minima procedens semper proxime ad veritatem accedat et, quousque libuerit, progredi liceat. Sequenti autem modo haec methodus magis perfici poterit.

PROBLEMA 86

656. *Methodum praecedentem aequationes differentiales proxime integrandi magis perficere, ut minus a veritate aberret.*

SOLUTIO

Proposita aequatione integranda $\frac{dy}{dx} = V$ error methodi supra expositae inde oritur, quod per singula intervalla functio V ut constans spectetur, cum tamen revera mutationem subeat, praecipue nisi intervalla statuantur minima. Variabilitas autem ipsius V per quodvis intervallum simili modo in computum duci potest, quo in sectione praecedente § 321 usi sumus. Scilicet si iam ipsi x conveniat y, ex natura differentialium ipsi $x - ndx$ vidimus convenire

$$y - ndy + \frac{n(n+1)}{1\cdot 2}ddy - \frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}d^3y + \text{etc.},$$

qui valor sumto n infinito erit

$$y - ndy + \frac{nnddy}{1\cdot 2} - \frac{n^3d^3y}{1\cdot 2\cdot 3} + \frac{n^4d^4y}{1\cdot 2\cdot 3\cdot 4} - \text{etc.}$$

Statuatur iam $x - ndx = a$ et $y - ndy + \frac{nnddy}{1\cdot 2} - \frac{n^3d^3y}{1\cdot 2\cdot 3} + \frac{n^4d^4y}{1\cdot 2\cdot 3\cdot 4} - \text{etc.} = b$ hique valores in quovis intervallo ut primi spectentur, dum extremi per x et y indicantur. Cum igitur sit $n = \frac{x-a}{dx}$, fiet

$$y = b + \frac{(x-a)dy}{dx} - \frac{(x-a)^2ddy}{1\cdot 2\,dx^2} + \frac{(x-a)^3d^3y}{1\cdot 2\cdot 3\,dx^3} - \frac{(x-a)^4d^4y}{1\cdot 2\cdot 3\cdot 4\,dx^4} + \text{etc.},$$

quae expressio, si x non multum superat a, valde convergit ideoque admodum est idonea ad valorem y proxime inveniendum. Verum ad singulos terminos huius seriei evolvendos notari oportet esse $\frac{dy}{dx} = V$ hincque $\frac{ddy}{dx^2} = \frac{dV}{dx}$. Cum autem V sit functio ipsarum x et y, si ponamus $dV = Mdx + Ndy$, ob $\frac{dy}{dx} = V$ erit $\frac{ddy}{dx^2} = M + NV$ seu exprimendi modo iam supra exposito

$$\frac{ddy}{dx^2} = \left(\frac{dV}{dx}\right) + V\left(\frac{dV}{dy}\right);$$

quae expressio uti nata est ex praecedente $\frac{dy}{dx} = V$, ita ex ea nascetur sequens

$$\frac{d^3y}{dx^3} = \left(\frac{ddV}{dx^2}\right) + \left(\frac{dV}{dx}\right)\left(\frac{dV}{dy}\right) + 2V\left(\frac{ddV}{dx\,dy}\right) + V\left(\frac{dV}{dy}\right)^2 + VV\left(\frac{ddV}{dy^2}\right).$$

Quoniam vero ipse valor ipsius y nondum est cognitus, hoc modo saltem obtinetur aequatio algebraica, qua relatio inter x et y exprimitur, nisi forte sufficiat in terminis minimis posuisse $y = b$.

Altera autem operatio § 322 exposita valorem ipsius y, qui ipsi x in fine cuiusque intervalli respondet, explicite determinabit, cum in initio eiusdem intervalli fuerit $x = a$ et $y = b$. Cum enim hinc posito $x = a + nda$, siquidem a et b ut variabiles spectemus, fiat

$$y = b + ndb + \frac{n(n-1)}{1 \cdot 2} ddb + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} d^3 b + \text{etc.},$$

quia est $n = \frac{x-a}{da}$ ideoque numerus infinitus, erit

$$y = b + \frac{(x-a)\,db}{da} + \frac{(x-a)^2\,ddb}{1 \cdot 2\,da^2} + \frac{(x-a)^3\,d^3b}{1 \cdot 2 \cdot 3\,da^3} + \text{etc.}$$

Est vero $\frac{db}{da} = V$, siquidem in functione V scribatur $x = a$ et $y = b$; tum vero iisdem pro x et y valoribus substitutis erit

$$\frac{ddb}{da^2} = \left(\frac{dV}{dx}\right) + V\left(\frac{dV}{dy}\right)$$

et

$$\frac{d^3b}{da^3} = \left(\frac{ddV}{dx^2}\right) + 2V\left(\frac{ddV}{dx\,dy}\right) + VV\left(\frac{ddV}{dy^2}\right) + \left(\frac{dV}{dy}\right)\left(\left(\frac{dV}{dx}\right) + V\left(\frac{dV}{dy}\right)\right),$$

unde sequentes simili modo formari oportet. Sit igitur, postquam scripserimus $x = a$ et $y = b$,

$$\frac{dy}{dx} = A, \quad \frac{ddy}{dx^2} = B, \quad \frac{d^3y}{dx^3} = C, \quad \frac{d^4y}{dx^4} = D \quad \text{etc.}$$

ac valori $x = a + \omega$ conveniet iste valor

$$y = b + A\omega + \frac{1}{2} B\omega^2 + \frac{1}{6} C\omega^3 + \frac{1}{24} D\omega^4 + \text{etc.},$$

qui duo valores iam pro sequente intervallo erunt initiales, ex quibus simili modo finales erui oportet.

COROLLARIUM 1

657. Quoniam hic variabilitatis functionis V rationem habuimus, intervalla iam maiora statuere licet, ac si illas formulas A, B, C, D etc. in infinitum continuare vellemus, intervalla quantumvis magna assumi possent; tum autem pro y oriretur series infinita.

COROLLARIUM 2

658. Si seriei inventae tantum binos terminos primos sumamus, ut sit $y = b + A\omega$, habebitur determinatio praecedens, unde simul patet errorem ibi commissum sequentibus terminis iunctim sumtis aequari.

COROLLARIUM 3

659. Etiamsi autem seriei inventae plures terminos capiamus, consultum tamen non erit intervalla nimis magna constitui, ut ω valorem modicum obtineat, praecipue si quantitates B, C, D etc. evadant valde magnae.

SCHOLION

660. Maximo incommodo hae operationes turbantur, si quando horum coefficientium A, B, C, D etc. quidam in infinitum excrescant. Evenit autem hoc tantum in certis intervallis, ubi ipsa quantitas V vel in nihilum abit vel in infinitum, cui incommodo quemadmodum sit occurrendum, iam innuimus et mox accuratius ostendemus. Caeterum calculus pro singulis intervallis pari modo instituitur, ita ut, cum eius ratio pro intervallo primo fuerit inventa, quod incipit a valoribus pro lubitu assumtis $x = a$ et $y = b$, eadem pro sequentibus intervallis sit valitura. Cum enim pro fine intervalli primi fiat $x = a + \omega = a'$ et $y = b + A\omega + \frac{1}{2}B\omega^2 + \frac{1}{6}C\omega^3 + \frac{1}{24}D\omega^4 + \text{etc.} = b'$, hi erunt valores initiales pro intervallo secundo, ex quibus simili modo finales elici oportet; hic scilicet calculus innitetur perinde litteris a' et b' ac prior litteris a et b, id quod clarius ex exemplis subiunctis patebit.

EXEMPLUM 1

661. *Aequationis differentialis* $dy = dx(x^n + cy)$ *integrale completum proxime investigare.*

Cum hic sit $V = \frac{dy}{dx} = x^n + cy$, erit differentiando $\frac{ddy}{dx^2} = nx^{n-1} + cx^n + ccy$ sicque porro

$$\frac{d^3y}{dx^3} = n(n-1)x^{n-2} + ncx^{n-1} + ccx^n + c^3y,$$

$$\frac{d^4y}{dx^4} = n(n-1)(n-2)x^{n-3} + n(n-1)cx^{n-2} + nccx^{n-1} + c^3x^n + c^4y$$

etc.

Quodsi ergo ponamus valori $x = a$ convenire $y = b$, alii cuicunque valori $x = a + \omega$ conveniet

$$y = b + \omega(a^n + cb) + \tfrac{1}{2}\omega^2(ccb + ca^n + na^{n-1})$$
$$+ \tfrac{1}{6}\omega^3\left(c^3b + cca^n + nca^{n-1} + n(n-1)a^{n-2}\right)$$
$$+ \tfrac{1}{24}\omega^4\left(c^4b + c^3a^n + ncca^{n-1} + n(n-1)ca^{n-2} + n(n-1)(n-2)a^{n-3}\right)$$
$$\text{etc.},$$

quae series sumta quantitate ω satis parva quantumvis promte convergit, sicque posito $a + \omega = a'$ et respondente valore ipsius $y = b'$ hinc simili modo ad sequentes perveniemus, quam operationem, quousque lubuerit, continuare licet.

EXEMPLUM 2

662. *Aequationis differentialis $dy = dx(xx + yy)$ integrale completum proxime investigare.*

Cum hic sit $\frac{dy}{dx} = V = xx + yy$, erit continuo differentiando

$$\frac{ddy}{dx^2} = 2x + 2xxy + 2y^3$$

et

$$\frac{d^3y}{dx^3} = 2 + 4xy + 2x^4 + 8xxyy + 6y^4,$$

$$\frac{d^4y}{dx^4} = 4y + 12x^3 + 20xyy + 16x^4y + 40xxy^3 + 24y^5,$$

$$\frac{d^5y}{dx^5} = 40x^2 + 24y^2 + 104x^3y + 120xy^3 + 16x^6 + 136x^4y^2 + 240x^2y^4 + 120y^6$$
$$\text{etc.}$$

Quare si initio sit $x = a$ et $y = b$, erit

$A = aa + bb,$

$B = 2a + 2aab + 2b^3,$

$C = 2 + 4ab + 2a^4 + 8aabb + 6b^4,$

$D = 4b + 12a^3 + 20abb + 16a^4b + 40aab^3 + 24b^5,$

$E = 40a^2 + 24b^2 + 104a^3b + 120ab^3 + 16a^6 + 136a^4b^2 + 240a^2b^4 + 120b^6,$

unde valori cuicunque alii $x = a + \omega$ conveniet

$$y = b + A\omega + \tfrac{1}{2}B\omega^2 + \tfrac{1}{6}C\omega^3 + \tfrac{1}{24}D\omega^4 + \tfrac{1}{120}E\omega^5 + \text{etc.},$$

atque ex talibus binis valoribus, qui sint $x = a'$ et $y = b'$, denuo sequentes elici possunt.

SCHOLION

663. Quoniam totum negotium ad inventionem horum coefficientium A, B, C, D etc. redit, observo eosdem sine differentiatione inveniri posse, id quod in hoc postremo exemplo $\frac{dy}{dx} = xx + yy$ ita praestabitur. Cum statuamus posito $x = a$ fieri $y = b$, ponamus in genere $x = a + \omega$ et $y = b + \psi$ et nostra aequatio induet hanc formam

$$\frac{d\psi}{d\omega} = aa + bb + 2a\omega + \omega\omega + 2b\psi + \psi\psi,$$

et quia evanescente ω simul evanescit ψ, sumamus

$$\psi = \alpha\omega + \beta\omega^2 + \gamma\omega^3 + \delta\omega^4 + \varepsilon\omega^5 + \text{etc.}$$

hocque valore substituto prodibit

$$\begin{aligned} \alpha + 2\beta\omega + 3\gamma\omega^2 + 4\delta\omega^3 + 5\varepsilon\omega^4 + \text{etc.} \\ = aa + bb + 2a\omega + \omega^2 \\ + 2\alpha b\omega + 2\beta b\omega^2 + 2\gamma b\omega^3 + 2\delta b\omega^4 \\ + \alpha^2\omega^2 + 2\alpha\beta\omega^3 + 2\alpha\gamma\omega^4 \\ + \beta\beta\omega^4 \end{aligned}$$

Singulis ergo terminis ad nihilum reductis fiet

$$\alpha = aa + bb, \quad 2\beta = 2\alpha b + 2a, \quad 3\gamma = 2\beta b + \alpha\alpha + 1, \quad 4\delta = 2\gamma b + 2\alpha\beta,$$
$$5\varepsilon = 2\delta b + 2\alpha\gamma + \beta\beta, \quad 6\zeta = 2\varepsilon b + 2\alpha\delta + 2\beta\gamma \quad \text{etc.},$$

unde iidem valores qui supra per differentiationem eliciuntur. Uti haec methodus simplicior est praecedente, ita etiam hoc illi praestat, quod semper in usum vocari possit, cum illa interdum frustra applicetur, veluti in exemplis allatis evenit, si valores initiales a et b evanescant, ubi plerique coefficientes in nihilum abirent. Quod idem incommodum iam supra animadvertimus, cum adeo evenire possit, ut omnes coefficientes vel evanescant vel in infinitum abeant. Verum hoc nonnisi in certis intervallis usu venit, pro quibus ergo calculum peculiari modo institui conveniet; reliquis autem intervallis methodus hic exposita per differentiationem procedens commodius adhiberi videtur, quippe quae saepe facilius instituitur quam substitutio certisque regulis continetur semper locum habentibus etiam in aequationibus transcendentibus. Quare pro singularibus illis intervallis praecepta tradere oportebit.

PROBLEMA 87

664. *Si in integratione aequationis $\frac{dy}{dx} = V$ pro quopiam intervallo eveniat, ut quantitas V vel evanescat vel fiat infinita, integrationem pro isto intervallo instituere.*

SOLUTIO

Sit pro initio intervalli, quod contemplamur, $x = a$ et $y = b$; quo casu cum V vel evanescat vel in infinitum abeat, ponamus $\frac{dy}{dx} = \frac{P}{Q}$, ita ut posito $x = a$ et $y = b$ vel P vel Q vel utrumque evanescat. Statuamus ergo, ut ab his terminis ulterius progrediamur, $x = a + \omega$ et $y = b + \psi$ fietque $\frac{dy}{dx} = \frac{d\psi}{d\omega}$ atque tam P quam Q erit functio ipsarum ω et ψ, quarum altera saltem evanescat facto $\omega = 0$ et $\psi = 0$. Iam ad rationem inter ω et ψ proxime saltem investigandam ponatur $\psi = m\omega^n$; erit $\frac{d\psi}{d\omega} = mn\omega^{n-1}$ hincque $mnQ\omega^{n-1} = P$, ubi P et Q ob $\psi = m\omega^n$ meras potestates ipsius ω continebunt, quarum tantum minimas in calculo retinuisse sufficit, cum altiores prae his ut evanescentes spectari queant. Infimae ergo potestates ipsius ω inter se aequales reddantur simulque ad nihilum redigantur; unde tam exponens n quam coefficiens m determinabitur. Si deinde relationem inter ω et ψ exactius cognoscere velimus, inventis m et n ad altiores potestates ascendamus ponendo $\psi = m\omega^n + M\omega^{n+\mu} + N\omega^{n+\nu} +$ etc. hincque simili modo sequentes partes definientur, quousque ob magnitudinem intervalli seu particulae ω necessarium visum fuerit.

COROLLARIUM 1

665. Si posito $x = a$ et $y = b$ neque P neque Q evanescat, substitutione adhibita reperietur $\frac{d\psi}{d\omega} = \frac{A + \text{etc.}}{\alpha + \text{etc.}}$ hincque proxime $\alpha d\psi = A d\omega$ et $\psi = \frac{A}{\alpha}\omega$, qui est primus terminus praecedentis approximationis, quo invento reliqui ut ante se habebunt.

COROLLARIUM 2

666. Si α tantum evanescat, habebitur $\frac{d\psi}{d\omega}(M\omega^\mu + N\psi^\nu) = A$ proxime, unde posito $\psi = m\omega^n$ fit $A = mn\omega^{n-1}(M\omega^\mu + Nm^\nu\omega^{n\nu})$; ubi si $n\nu > \mu$, debet esse $n = 1 - \mu$ et $mnM = A$; quod autem non valet, nisi sit $\nu(1 - \mu) > \mu$

seu $\nu > \frac{\mu}{1-\mu}$. Sin autem sit $\nu < \frac{\mu}{1-\mu}$, statui debet $n-1+n\nu=0$ seu $n=\frac{1}{1+\nu}$ altero termino ut infima potestate spectato. At si fuerit $\nu = \frac{\mu}{1-\mu}$, ambo termini pro paribus potestatibus erunt habendi fietque $n=1-\mu$ et $A=mn(M+Nm^{\nu})$, unde m definiri debet.

SCHOLION

667. In genere hic vix quicquam praecipere licet, sed quovis casu oblato haud difficile est omnia, quae ad solutionem perducunt, perspicere. Siquidem omnes exponentes essent integri, regula illa NEUTONIANA, qua ope parallelogrammi resolutio aequationum instruitur, hic in usum vocari posset; tum vero exponentium fractorum ad integros reductio satis est nota. Verum huiusmodi casus tam raro occurrunt, ut inutile foret in praeceptis prolixum esse, quae quovis casu ab exercitato facile conduntur. Veluti si perveniatur ad hanc aequationem $\frac{d\psi}{d\omega}(\alpha\sqrt{\omega}+\beta\psi)=\gamma$, ex superioribus patet primam operationem dare $\psi=m\sqrt{\omega}$, unde fit $\frac{1}{2}m(\alpha+\beta m)=\gamma$, unde m innotescit idque duplici modo. Quin etiam haec aequatio posito $\sqrt{\omega}=p$ ad homogeneitatem reducitur ideoque revera integrari potest. Verum haec vix unquam usum habitura fusius non prosequor, sed, quod adhuc in hac parte pertractandum restat, exponam, quomodo eiusmodi aequationes differentiales resolvi oporteat, in quibus differentialium ratio, puta $\frac{dy}{dx}=p$, vel plures obtinet dimensiones vel adeo transcendenter ingreditur; quo absoluto partem secundam, in qua differentialia altiorum graduum occurrunt, aggrediar.

CALCVLI INTEGRALIS
LIBER PRIOR.

PARS PRIMA

SEV

METHODVS INVESTIGANDI FVNCTIONES VNIVS VARIABILIS EX DATA RELATIONE QVACVNQVE DIFFERENTIALIVM PRIMI GRADVS.

SECTIO TERTIA

DE

RESOLVTIONE AEQVATIONVM DIFFERENTIALIVM MAGIS COMPLICATARVM.

DE RESOLUTIONE AEQUATIONUM DIFFERENTIALIUM IN QUIBUS DIFFERENTIALIA AD PLURES DIMENSIONES ASSURGUNT VEL ADEO TRANSCENDENTER IMPLICANTUR

PROBLEMA 88

668. *Posita differentialium relatione* $\frac{dy}{dx}=p$ *si proponatur aequatio quaecunque inter binas quantitates* x *et* p, *relationem inter ipsas variabiles* x *et* y *investigare.*

SOLUTIO

Cum detur aequatio inter p et x, concessa aequationum resolutione ex ea quaeratur p per x ac reperietur functio ipsius x, quae ipsi p erit aequalis. Pervenietur ergo ad huiusmodi aequationem $p=X$ existente X functione quapiam ipsius x tantum. Quare cum sit $p=\frac{dy}{dx}$, habebimus $dy=Xdx$ sicque quaestio ad sectionem primam est reducta, unde formulae Xdx integrale investigari oportet; quo facto integrale quaesitum erit $y=\int Xdx$.

Si aequatio inter x et p data ita fuerit comparata, ut inde facilius x per p definiri possit, quaeratur x prodeatque $x=P$ existente P functione quadam ipsius p. Hac igitur aequatione differentiata erit $dx=dP$ hincque $dy=pdx=pdP$, unde integrando elicitur $y=\int pdP$ seu $y=pP-\int Pdp$. Hinc ergo ambae variabiles x et y per tertiam p ita determinantur, ut sit $x=P$ et $y=pP-\int Pdp$, unde relatio inter x et y est manifesta.

Si neque p commode per x neque x per p definiri queat, saepe effici potest, ut utraque commode per novam quantitatem u definiatur; ponamus ergo inveniri $x = U$ et $p = V$, ut U et V sint functiones eiusdem variabilis u. Hinc ergo erit $dy = pdx = VdU$ et $y = \int VdU$ sicque x et y per eandem novam variabilem u exprimuntur.

COROLLARIUM 1

669. Simili modo resolvetur casus, quo aequatio quaecunque inter p et alteram variabilem y proponitur, quoniam binas variabiles x et y inter se permutare licet. Tum autem, sive p per y sive y per p sive utraque per novam variabilem u definiatur, notari oportet esse $dx = \frac{dy}{p}$.

COROLLARIUM 2

670. Cum $\sqrt{(dx^2 + dy^2)}$ exprimat elementum arcus curvae, cuius coordinatae rectangulae sunt x et y, si ratio

$$\frac{\sqrt{(dx^2 + dy^2)}}{dx} = \sqrt{(1 + pp)} \quad \text{seu} \quad \frac{\sqrt{(dx^2 + dy^2)}}{dy} = \frac{\sqrt{(1 + pp)}}{p}$$

aequetur functioni vel ipsius x vel ipsius y, hinc relatio inter x et y inveniri poterit.

COROLLARIUM 3

671. Quoniam hoc modo relatio inter x et y per integrationem invenitur, simul nova quantitas constans introducitur, quocirca illa relatio pro integrali completo erit habenda.

SCHOLION 1

672. Hactenus eiusmodi tantum aequationes differentiales examini subiecimus, quibus posito $\frac{dy}{dx} = p$ eiusmodi relatio inter ternas quantitates x, y et p proponitur, unde valor ipsius p commode per x et y exprimi potest, ita ut $p = \frac{dy}{dx}$ aequetur functioni cuipiam ipsarum x et y. Nunc igitur eiusmodi relationes inter x, y et p considerandae veniunt, ex quibus valorem ipsius p vel minus commode vel plane non per x et y definire liceat; atque hic simplicissimus casus sine dubio est, quando in relatione proposita altera variabilis x seu y plane deest, ita ut tantum relatio inter p et x vel p et y proponatur; quem casum in hoc problemate expedivimus.

Solutionis autem vis in eo versatur, ut proposita aequatione inter x et p non littera p per x, nisi forte hoc facile praestari queat, sed potius x per p vel etiam utraque per novam variabilem u definiatur. Veluti si proponatur haec aequatio

$$xdx + ady = b\sqrt{(dx^2 + dy^2)},$$

quae posito $\frac{dy}{dx} = p$ abit in hanc $x + ap = b\sqrt{(1 + pp)}$, hinc minus commode definiretur p per x. Cum autem sit

$$x = b\sqrt{(1 + pp)} - ap,$$

ob $y = \int pdx = px - \int xdp$ erit

$$y = bp\sqrt{(1 + pp)} - app - b\int dp\sqrt{(1 + pp)} + \tfrac{1}{2}app$$

sicque relatio inter x et y constat.

Sin autem perventum fuerit ad talem aequationem

$$x^3dx^3 + dy^3 = axdx^2dy \quad \text{seu} \quad x^3 + p^3 = apx,$$

hinc neque x per p neque p per x commode definire licet; ex quo pono $p = ux$, unde fit $x + u^3x = au$ hincque

$$x = \frac{au}{1 + u^3} \quad \text{et} \quad p = \frac{auu}{1 + u^3}.$$

Iam ob $dx = \frac{adu(1 - 2u^3)}{(1 + u^3)^2}$ colligitur

$$y = aa\int \frac{uudu(1 - 2u^3)}{(1 + u^3)^3}$$

ac reducendo hanc formam ad simpliciorem

$$y = \frac{1}{6}aa\frac{2u^3 - 1}{(1 + u^3)^2} - aa\int \frac{uudu}{(1 + u^3)^2}$$

seu

$$y = \frac{1}{6}aa\frac{2u^3 - 1}{(1 + u^3)^2} + \frac{1}{3}aa\frac{1}{1 + u^3} + \text{Const.}$$

SCHOLION 2

673. Cum igitur hunc casum, quo aequatio vel inter x et p vel inter y et p proponitur, generatim expedire licuerit, videndum est, quibus casibus

evolutio succedat, quando omnes tres quantitates x, y et p in aequatione proposita insunt. Ac primo quidem observo, dummodo binae variabiles x et y ubique eundem dimensionum numerum adimpleant, quomodocunque praeterea quantitas p ingrediatur, resolutionem semper ad casus ante tractatos revocari posse; tales scilicet aequationes perinde tractare licet atque aequationes homogeneas, ad quod genus etiam merito referuntur, cum dimensiones a differentialibus natae ubique debeant esse pares et iudicium ex solis quantitatibus finitis x et y peti oporteat. Quae ergo dummodo ubique eundem dimensionum numerum constituant, aequatio pro homogenea erit habenda, veluti est $xxdy - yy\sqrt{(dx^2 + dy^2)} = 0$ seu $pxx - yy\sqrt{(1 + pp)} = 0$. Deinde etiam eiusmodi aequationes evolutionem admittunt, in quibus altera variabilis x vel y plus una dimensione nusquam habet, utcunque praeterea differentialium ratio $p = \frac{dy}{dx}$ ingrediatur. Hos ergo casus hic accuratius explicemus.

PROBLEMA 89

674. *Posito* $p = \frac{dy}{dx}$ *si in aequatione inter* x, y *et* p *proposita binae variabiles* x *et* y *ubique eundem dimensionum numerum compleant, invenire relationem inter* x *et* y, *quae illius aequationis sit integrale completum.*

SOLUTIO

Cum in aequatione inter x, y et p proposita binae variabiles x et y ubique eundem dimensionum numerum constituant, si ponamus $y = ux$, quantitas x inde per divisionem tolletur habebiturque aequatio inter duas tantum quantitates u et p, qua earum relatio ita definietur, ut vel u per p vel p per u determinari possit. Iam ex positione $y = ux$ sequitur $dy = udx + xdu$; cum igitur sit $dy = pdx$, erit $pdx - udx = xdu$ ideoque $\frac{dx}{x} = \frac{du}{p - u}$. Quia itaque p per u datur, formula differentialis $\frac{du}{p - u}$ unicam variabilem complectens per regulas primae sectionis integretur eritque $lx = \int \frac{du}{p - u}$ sicque x per u determinatur; et cum sit $y = ux$, ambae variabiles x et y per eandem tertiam variabilem u determinantur, et quia illa integratio constantem arbitrariam inducit, haec relatio inter x et y erit integrale completum.

COROLLARIUM 1

675. Cum sit $\frac{dx}{x} = \frac{du}{p-u}$, erit etiam $lx = -l(p-u) + \int \frac{dp}{p-u}$, quae formula commodior est, si forte ex aequatione inter p et u proposita quantitas u facilius per p definitur.

COROLLARIUM 2

676. Quodsi integrale $\int \frac{du}{p-u}$ vel $\int \frac{dp}{p-u}$ per logarithmos exprimi possit, ut sit $\int \frac{du}{p-u} = lU$, erit $lx = lC + lU$ hincque $x = CU$ et $y = CUu$; unde relatio inter x et y algebraice dabitur, et cum sit $u = \frac{y}{x}$, haec tertia variabilis u facile eliditur.

SCHOLION

677. Eandem hanc resolutionem supra in aequationibus homogeneis ordinariis docuimus, quae ergo ob dimensiones differentialium non turbatur; quin etiam succedit, etiamsi ratio differentialium $\frac{dy}{dx} = p$ transcendenter ingrediatur. Hoc modo scilicet resolutio ad integrationem aequationis differentialis separatae $\frac{dx}{x} = \frac{du}{p-u}$ perducitur, quemadmodum etiam supra per priorem methodum negotium fuit expeditum. Altera vero methodus, qua supra usi sumus quaerendo factorem, qui aequationem differentialem reddat per se integrabilem, hic plane locum non habet, cum per differentiationem aequationis finitae nunquam differentialia ad plures dimensiones exsurgere queant. Non ergo hoc modo invenitur aequatio finita inter x et y, quae differentiata ipsam aequationem propositam reproducat, sed quae saltem cum ea conveniat et quidem non obstante arbitraria illa constante, quae per integrationem ingressa integrale completum reddit.

EXEMPLUM 1

678. *Si in aequationem propositam neutra variabilium x et y ipsa ingrediatur, sed tantum differentialium ratio $\frac{dy}{dx} = p$, integrale completum assignare.*

Posito ergo $\frac{dy}{dx} = p$ aequatio proposita solam variabilem p cum constantibus complectetur, unde ex eius resolutione, prout plures involvat radices, orietur $p = \alpha$, $p = \beta$, $p = \gamma$ etc. Iam ob $p = \frac{dy}{dx}$ ex singulis radicibus integralia completa elicientur, quae erunt

$$y = \alpha x + a, \quad y = \beta x + b, \quad y = \gamma x + c \quad \text{etc.},$$

quae singula aequationi propositae aeque satisfaciunt. Quae si velimus omnia una aequatione finita complecti, erit integrale completum

$$(y - \alpha x - a)(y - \beta x - b)(y - \gamma x - c) \text{ etc.} = 0,$$

quae, uti apparet, non unam novam constantem, sed plures a, b, c etc. comprehendit, tot scilicet, quot aequatio differentialis plurium dimensionum habuerit radices.

COROLLARIUM 1

679. Ita aequationis differentialis

$$dy^2 - dx^2 = 0 \quad \text{seu} \quad pp - 1 = 0$$

ob $p = +1$ et $p = -1$ duo habemus integralia $y = x + a$ et $y = -x + b$, quae in unum collecta dant $(y - x - a)(y + x - b) = 0$ seu

$$yy - xx - (a + b)y - (a - b)x + ab = 0.$$

COROLLARIUM 2

680. Proposita aequatione

$$dy^3 + dx^3 = 0 \quad \text{seu} \quad p^3 + 1 = 0$$

ob radices $p = -1$, $p = \frac{1 + \sqrt{-3}}{2}$ et $p = \frac{1 - \sqrt{-3}}{2}$ erit vel

$$y = -x + a \quad \text{vel} \quad y = \frac{1 + \sqrt{-3}}{2} x + b \quad \text{vel} \quad y = \frac{1 - \sqrt{-3}}{2} x + c,$$

quae collecta praebent

$$y^3 + x^3 - (a+b+c)yy + \left(a - \frac{1+\sqrt{-3}}{2} b - \frac{1-\sqrt{-3}}{2} c\right) xy + \left(-a + \frac{1-\sqrt{-3}}{2} b + \frac{1+\sqrt{-3}}{2} c\right) xx$$

$$+ (ab + ac + bc)y + \left(bc - \frac{1 - \sqrt{-3}}{2} ac - \frac{1 + \sqrt{-3}}{2} ab\right) x - abc = 0,$$

quae aequatio etiam ita exhiberi potest

$$y^3 + x^3 - fyy - gxy - hxx + Ay + Bx + C = 0,$$

ubi constantes A, B, C ita debent esse comparatae, ut aequatio haec resolutionem in tres simplices admittat.

EXEMPLUM 2

681. *Proposita aequatione differentiali* $ydx - x\sqrt{(dx^2 + dy^2)} = 0$ *eius integrale completum invenire.*

Posito $\frac{dy}{dx} = p$ fit $y - x\sqrt{(pp+1)} = 0$; sit ergo $y = ux$; erit

$$u = \sqrt{(pp+1)} \quad \text{et} \quad \frac{dx}{x} = \frac{du}{p-u},$$

unde per alteram formulam

$$lx = -l(p-u) + \int \frac{dp}{p - \sqrt{(pp+1)}} = -l(p-u) - \int dp\,(p + \sqrt{(pp+1)}),$$

at

$$\int dp\sqrt{(pp+1)} = \frac{1}{2}p\sqrt{(1+pp)} + \frac{1}{2}l(p + \sqrt{(1+pp)}),$$

unde colligitur

$$lx = C - \frac{1}{2}l(\sqrt{(1+pp)} - p) - \frac{1}{2}p\sqrt{(1+pp)} - \frac{1}{2}pp$$

$$= C + \frac{1}{2}l(\sqrt{(1+pp)} + p) - \frac{1}{2}p\sqrt{(1+pp)} - \frac{1}{2}pp$$

et

$$y = ux = x\sqrt{(pp+1)}.$$

EXEMPLUM 3

682. *Huius aequationis* $ydx - xdy = nx\sqrt{(dx^2 + dy^2)}$ *integrale completum invenire.*

Ob $\frac{dy}{dx} = p$ nostra aequatio est $y - px = nx\sqrt{(1+pp)}$, quae posito $y = ux$ abit in $u - p = n\sqrt{(1+pp)}$. Cum ergo sit

$$lx = -l(p-u) + \int \frac{dp}{p-u},$$

erit

$$lx = -ln\sqrt{(1+pp)} - \int \frac{dp}{n\sqrt{(1+pp)}}$$

hincque

$$lx = C - ln\sqrt{(1+pp)} - \frac{1}{n}l(p+\sqrt{(1+pp)}).$$

Quare habetur

$$x = \frac{a}{\sqrt{(1+pp)}}(\sqrt{(1+pp)}-p)^{\frac{1}{n}} \quad \text{et} \quad y = \frac{a(p+n\sqrt{(1+pp)})}{\sqrt{(1+pp)}}(\sqrt{(1+pp)}-p)^{\frac{1}{n}}.$$

Cum nunc sit $uu - 2up + pp = nn + nnpp$, erit

$$p = \frac{u - n\sqrt{(uu+1-nn)}}{1-nn} \quad \text{et} \quad \sqrt{(1+pp)} = \frac{-nu+\sqrt{(uu+1-nn)}}{1-nn}$$

atque

$$\sqrt{(1+pp)} - p = \frac{-u+\sqrt{(uu+1-nn)}}{1-n},$$

unde fit

$$\frac{x(-nu+\sqrt{(uu+1-nn)})}{a(1-nn)} = \left(\frac{-u+\sqrt{(uu+1-nn)}}{1-n}\right)^{\frac{1}{n}},$$

ubi $u = \frac{y}{x}$. At si $n = 1$, erit $p = \frac{uu-1}{2u}$, $\sqrt{(1+pp)} = \frac{uu+1}{2u}$ atque

$$x = \frac{2au}{uu+1}\cdot\frac{1}{u} = \frac{2axx}{xx+yy} \quad \text{seu} \quad yy + xx = 2ax.$$

Si $n = -1$, est quidem ut ante $p = \frac{uu-1}{2u}$ et $\sqrt{(1+pp)} = \frac{-uu-1}{2u}$, unde

$$x = \frac{a}{\sqrt{(1+pp)}}\left(\sqrt{(1+pp)}+p\right) = \frac{2a}{1+uu} = \frac{2axx}{xx+yy}.$$

Ergo et $x = 0$ et $xx + yy - 2ax = 0$.[1])

SCHOLION

683. Haec aequatio sumendis utrinque quadratis et radice $p = \frac{dy}{dx}$ extrahenda ad aequationem homogeneam ordinariam reducitur. Fit enim primo

$$yy - 2pxy + ppxx = nnxx + nnppxx,$$

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1) Editio princeps: $x = \cdots = \frac{-2a}{1+uu} = \frac{-2axx}{xx+yy}$. *Ergo et* $x = 0$ *et* $xx + yy + 2ax = 0$.
Correxit F. E.

tum vero

$$px = \frac{xdy}{dx} = \frac{y \pm n\sqrt{(yy + xx - nnxx)}}{1 - nn},$$

quae posito $y = ux$ separabilis redditur. Ubi imprimis casus, quo $nn = 1$, notari meretur, quo fit $yy - 2pxy = xx$ seu $p = \frac{dy}{dx} = \frac{yy - xx}{2xy}$ ideoque

$$2xydy + xxdx - yydx = 0,$$

quae etiam per partes integrari potest, cum $2xydy - yydx$ integrabile fiat per factorem $\frac{1}{xyy}f : \frac{yy}{x}$; quo ut etiam pars $xxdx$ integrabilis reddatur, illa forma abit in $\frac{1}{xx}$ sicque habebitur $\frac{2xydy - yydx}{xx} + dx = 0$, cuius integrale est $\frac{yy}{x} + x = 2a$ ut ante, nisi quod altera solutio $x = 0$ hinc non eliciatur. Verum cum aequatio illa quadrata posito $n = 1$ subito abeat in simplicem, altera radix perit, quae reperitur ponendo $n = 1 - \alpha$, quo fit

$$yy - 2pxy = xx - 2\alpha xx - 2\alpha ppxx$$

ideoque px infinitum; reiectis ergo terminis prae reliquis evanescentibus est $-2pxy = xx - 2\alpha ppxx$, quae divisibilis per x alteram praebet solutionem $x = 0$. Talis quidem resolutio succedit, quando valorem p per radicis extractionem elicere licet; sed si aequatio ad plures dimensiones ascendat vel adeo transcendens fiat, methodo hic exposita carere non possumus.

EXEMPLUM 4

684. *Proposita aequatione* $xdy^3 + ydx^3 = dydx\sqrt{xy(dx^2 + dy^2)}$ *eius integrale completum investigare.*

Posito $\frac{dy}{dx} = p$ et $y = ux$ nostra aequatio induet hanc formam

$$p^3 + u = p\sqrt{u(1 + pp)},$$

unde conficitur

$$\frac{dx}{x} = \frac{du}{p - u} \quad \text{seu} \quad lx = \int \frac{du}{p - u} = -l(p - u) + \int \frac{dp}{p - u}.$$

Inde autem est

$$\sqrt{u} = \frac{1}{2}p\sqrt{(1 + pp)} + \frac{1}{2}p\sqrt{(1 - 4p + pp)}$$

et quadrando

$$u = \frac{1}{2}pp - p^3 + \frac{1}{2}p^4 + \frac{1}{2}pp\sqrt{(1+pp)(1-4p+pp)}$$

hincque

$$p - u = \frac{1}{2}p(1+pp)(2-p) - \frac{1}{2}pp\sqrt{(1+pp)(1-4p+pp)},$$

unde colligimus

$$\frac{dp}{p-u} = \frac{dp(2-p)}{2p(1-p+pp)} + \frac{dp\sqrt{(1-4p+pp)}}{2(1-p+pp)\sqrt{(1+pp)}}.$$

In quorum membrorum posteriore si ponatur $\sqrt{\frac{1-4p+pp}{1+pp}} = q$, ob

$$p = \frac{2+\sqrt{(4-(1-qq)^2)}}{1-qq}, \quad dp = \frac{4q\,dq(2+\sqrt{(4-(1-qq)^2)})}{(1-qq)^2\sqrt{(4-(1-qq)^2)}}$$

et

$$1 - p + pp = \frac{(3+qq)(2+\sqrt{(4-(1-qq)^2)})}{(1-qq)^2}$$

obtinebitur

$$\int\frac{dp}{p-u} = \frac{1}{2}\int\frac{dp(2-p)}{p(1-p+pp)} + 2\int\frac{qq\,dq}{(3+qq)\sqrt{(4-(1-qq)^2)}},$$

ubi membrum posterius neque per logarithmos neque arcus circulares integrari potest.

EXEMPLUM 5

685. *Invenire relationem inter x et y, ut posito $s = \int\sqrt{(dx^2+dy^2)}$ fiat $ss = 2xy$.*

Cum sit $s = \sqrt{2xy}$, erit

$$ds = \sqrt{(dx^2+dy^2)} = \frac{x\,dy + y\,dx}{\sqrt{2xy}}$$

hincque posito $\frac{dy}{dx} = p$ et $y = ux$ fiet

$$\sqrt{(1+pp)} = \frac{p+u}{\sqrt{2u}}$$

seu $u = \sqrt{2u(1+pp)} - p$ et radice extracta

$$\sqrt{u} = \sqrt{\frac{1+pp}{2}} + \frac{1-p}{\sqrt{2}} = \frac{1-p+\sqrt{(1+pp)}}{\sqrt{2}},$$

quare

$$u = 1 - p + pp + (1-p)\sqrt{(1+pp)} \quad \text{et} \quad p - u = -(1-p)(1-p+\sqrt{(1+pp)}).$$

Ergo

$$\int \frac{dp}{p-u} = \int \frac{dp}{2p(1-p)}(1 - p - \sqrt{(1+pp)}) = \frac{1}{2} lp - \frac{1}{2}\int \frac{dp\sqrt{(1+pp)}}{p(1-p)}.$$

At posito $p = \frac{1-qq}{2q}$ fit

$$\int \frac{dp\sqrt{(1+pp)}}{p(1-p)} = \int \frac{-dq(1+qq)^2}{q(1-qq)(qq+2q-1)} = \int \frac{dq}{q} - 2\int \frac{dq}{1-qq} - 4\int \frac{dq}{(q+1)^2-2}$$
$$= lq - l\frac{1+q}{1-q} + \sqrt{2}\, l\frac{\sqrt{2}+1+q}{\sqrt{2}-1-q}$$

hincque

$$\int \frac{dp}{p-u} = \frac{1}{2} lp - \frac{1}{2} lq + \frac{1}{2} l\frac{1+q}{1-q} - \frac{1}{\sqrt{2}} l\frac{\sqrt{2}+1+q}{\sqrt{2}-1-q} = l\left(\frac{1+q}{2q}\right) - \frac{1}{\sqrt{2}} l\frac{\sqrt{2}+1+q}{\sqrt{2}-1-q}.$$

Iam

$$p - u = \frac{(1+q)(1-2q-qq)}{2q} = \frac{(1+q)(2-(1+q)^2)}{2q}$$

sicque habetur

$$lx = C - l(1+q) + lq - l(2-(1+q)^2) + l\left(\frac{1+q}{q}\right) - \frac{1}{\sqrt{2}} l\frac{\sqrt{2}+1+q}{\sqrt{2}-1-q}$$
$$= l(2a) - l(2-(1+q)^2) - \frac{1}{\sqrt{2}} l\frac{\sqrt{2}+1+q}{\sqrt{2}-1-q},$$

ubi est $u = \frac{y}{x} = \frac{1}{2}(1+q)^2$ et $1 + q = \sqrt{\frac{2y}{x}}$, unde

$$x = \frac{ax}{x-y}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)^{\frac{1}{\sqrt{2}}} \quad \text{seu} \quad x - y = a\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)^{\frac{1}{\sqrt{2}}}$$

vel

$$(\sqrt{x}+\sqrt{y})^{1+\frac{1}{\sqrt{2}}} = a(\sqrt{x}-\sqrt{y})^{\frac{1}{\sqrt{2}}-1}.$$

Est ergo aequatio inter x et y interscendens, uti vocari solet.

SCHOLION

686. Facilius haec resolutio absolvitur quaerendo statim ex aequatione

$$u+p=\sqrt{2u(1+pp)} \quad \text{seu} \quad uu+2up+pp=2u+2upp$$

valorem ipsius p, qui fit

$$p=\frac{u+\sqrt{(uu-4uu+2u+2u^3-uu)}}{2u-1} \quad \text{seu} \quad p=\frac{u+(1-u)\sqrt{2u}}{2u-1}$$

et

$$p-u=\frac{(1-u)(2u+\sqrt{2u})}{2u-1}=\frac{(1-u)\sqrt{2u}}{\sqrt{2u}-1}.$$

Quare

$$lx=\int\frac{du}{p-u}=\int\frac{du(\sqrt{2u}-1)}{(1-u)\sqrt{2u}}=C-l(1-u)-\int\frac{du}{(1-u)\sqrt{2u}};$$

sit $u=vv$ eritque

$$\int\frac{du}{(1-u)\sqrt{2u}}=\frac{1}{\sqrt{2}}\int\frac{2dv}{1-vv}=\frac{1}{\sqrt{2}}\,l\frac{1+v}{1-v}$$

hincque

$$lx=la-l(1-u)-\frac{1}{\sqrt{2}}\,l\frac{1+\sqrt{u}}{1-\sqrt{u}}.$$

Unde ob $u=\frac{y}{x}$ reperitur

$$x=\frac{ax}{x-y}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)^{\frac{1}{\sqrt{2}}}$$

ut ante. Quare si curva desideretur coordinatis rectangulis x et y determinanda, ut eius arcus s sit $=\sqrt{2xy}$, erit aequatio eius naturam definiens

$$(\sqrt{x}+\sqrt{y})^{\frac{1}{\sqrt{2}}+1}=a\,(\sqrt{x}-\sqrt{y})^{\frac{1}{\sqrt{2}}-1}.$$

Caeterum evidens est simili modo quaestionem resolvi posse, si arcus s functioni cuicunque homogeneae unius dimensionis ipsarum x et y aequetur, seu si proponatur aequatio quaecunque homogenea inter x, y et s, id quod sequenti problemate ostendisse operae erit pretium.

PROBLEMA 90

687. *Si fuerit* $s = \int \sqrt{(dx^2 + dy^2)}$ *atque aequatio proponatur homogenea quaecunque inter* x, y *et* s, *in qua scilicet hae tres variabiles* x, y *et* s *ubique eundem dimensionum numerum constituant, invenire aequationem finitam inter* x *et* y.

SOLUTIO

Ponatur $y = ux$ et $s = vx$, ut hac substitutione ex aequatione homogenea proposita variabilis x elidatur et aequatio obtineatur inter binas u et v, unde v per u definiri possit. Tum vero sit $dy = pdx$ eritque $ds = dx\sqrt{(1+pp)}$, unde fit $pdx = udx + xdu$ et $dx\sqrt{(1+pp)} = vdx + xdv$, ergo

$$\frac{dx}{x} = \frac{du}{p-u} = \frac{dv}{\sqrt{(1+pp)} - v}.$$

Quia nunc v datur per u, sit $dv = qdu$, ut habeatur $\sqrt{(1+pp)} = v + pq - qu$ et sumtis quadratis $1 + pp = (v - qu)^2 + 2pq(v - qu) + ppqq$, unde elicitur

$$p = \frac{q(v-qu) + \sqrt{((v-qu)^2 - 1 + qq)}}{1-qq} \quad \text{et} \quad p - u = \frac{qv - u + \sqrt{((v-qu)^2 - 1 + qq)}}{1-qq}.$$

Quare hinc deducimus

$$\frac{dx}{x} = \frac{du(1-qq)}{qv - u + \sqrt{((v-qu)^2 - 1 + qq)}} = \frac{du(qv - u - \sqrt{((v-qu)^2 - 1 + qq)})}{1 + uu - vv},$$

unde, cum v et q detur per u, inveniri potest x per eandem u; at ob $qdu = dv$ fiet

$$lx = la - l\sqrt{(1 + uu - vv)} - \int \frac{du\sqrt{((v-qu)^2 - 1 + qq)}}{1 + uu - vv},$$

tum vero est $y = ux$ seu posito $\frac{y}{x}$ loco u habebitur aequatio quaesita inter x et y.

COROLLARIUM 1

688. Cum s exprimat arcum curvae coordinatis rectangulis x et y respondentem, sic definitur curva, cuius arcus aequatur functioni cuicunque

unius dimensionis ipsarum x et y; quae ergo erit algebraica, si integrale

$$\int \frac{du\sqrt{((v-qu)^2-1+qq)}}{1+uu-vv}$$

per logarithmos exhiberi potest.

COROLLARIUM 2

689. Simili modo resolvi poterit problema, si s eiusmodi formulam integralem exprimat, ut sit $ds = Qdx$ existente Q functione quacunque quantitatum p, u et v. Tum autem ex aequalitate $\frac{dx}{x} = \frac{du}{p-u} = \frac{dv}{Q-v}$ valorem ipsius p elici oportet, et quia v per u datur, erit $lx = \int \frac{du}{p-u}$.

EXEMPLUM 1

690. Si debeat esse $s = \alpha x + \beta y$, erit $v = \alpha + \beta u$ et $q = \frac{dv}{du} = \beta$, hinc $v - qu = \alpha$, ergo

$$lx = la - l\sqrt{\left(1 + uu - (\alpha + \beta u)^2\right)} - \int \frac{du\sqrt{(\alpha\alpha + \beta\beta - 1)}}{1 + uu - (\alpha + \beta u)^2},$$

quae postrema pars est

$$-\int \frac{du\sqrt{(\alpha\alpha + \beta\beta - 1)}}{1 - \alpha\alpha - 2\alpha\beta u + (1 - \beta\beta)uu} = (\alpha\alpha + \beta\beta - 1)^{\frac{1}{2}} \int \frac{du}{\alpha\alpha - 1 + 2\alpha\beta u + (\beta\beta - 1)uu},$$

quae transformatur in

$$\int \frac{(\beta\beta - 1)du\sqrt{(\alpha\alpha + \beta\beta - 1)}}{\left(u(\beta\beta - 1) + \alpha\beta - \sqrt{(\alpha\alpha + \beta\beta - 1)}\right)\left(u(\beta\beta - 1) + \alpha\beta + \sqrt{(\alpha\alpha + \beta\beta - 1)}\right)}$$
$$= \frac{1}{2} l \frac{(\beta\beta - 1)u + \alpha\beta - \sqrt{(\alpha\alpha + \beta\beta - 1)}}{(\beta\beta - 1)u + \alpha\beta + \sqrt{(\alpha\alpha + \beta\beta - 1)}}.$$

Quare posito $u = \frac{y}{x}$ aequatio integralis quaesita est sumtis quadratis

$$\frac{xx + yy - (\alpha x + \beta y)^2}{aa} = \frac{(\beta\beta - 1)y + \alpha\beta x - x\sqrt{(\alpha\alpha + \beta\beta - 1)}}{(\beta\beta - 1)y + \alpha\beta x + x\sqrt{(\alpha\alpha + \beta\beta - 1)}}.$$

At posito

$$(\beta\beta-1)y+\alpha\beta x-x\sqrt{(\alpha\alpha+\beta\beta-1)}=P,$$
$$(\beta\beta-1)y+\alpha\beta x+x\sqrt{(\alpha\alpha+\beta\beta-1)}=Q$$

est

$$PQ=(\beta\beta-1)^2yy+2\alpha\beta(\beta\beta-1)xy+(\alpha\alpha-1)(\beta\beta-1)xx$$
$$=(\beta\beta-1)((\alpha x+\beta y)^2-xx-yy),$$

unde mutata constante fit $\frac{PQ}{bb}=\frac{P}{Q}$, ergo vel $P=0$ vel $Q=b$; solutio ergo in genere est

$$(\beta\beta-1)y+\alpha\beta x\pm x\sqrt{(\alpha\alpha+\beta\beta-1)}=c,$$

quae est aequatio pro linea recta.

EXEMPLUM 2

691. Si debeat esse $s=\frac{nyy}{x}$, erit $v=nuu$ et $q=2nu$; unde

$$1+uu-vv=1+uu-nnu^4 \quad \text{et} \quad v-qu=-nuu,$$

ergo

$$lx=la-l\sqrt{(1+uu-nnu^4)}-\int\frac{du\sqrt{(nnu^4-1+4nnuu)}}{1+uu-nnu^4},$$

quae formula autem per logarithmos integrari nequit.

EXEMPLUM 3

692. Si debeat esse $ss=xx+yy$, erit $v=\sqrt{(1+uu)}$ et $q=\frac{u}{\sqrt{(1+uu)}}$, unde fit $1+uu-vv=0$; solutionem ergo ex primis formulis repeti convenit, unde fit

$$v-qu=\frac{1}{\sqrt{(1+uu)}}, \quad qq-1=\frac{-1}{1+uu} \quad \text{et} \quad qv-u=0;$$

ergo

$$p-u=0 \quad \text{seu} \quad \frac{dy}{dx}-\frac{y}{x}=0,$$

ita ut prodeat $y=nx$.

EXEMPLUM 4

693. Si debeat esse $ss=yy+nxx$ seu $v=\sqrt{(uu+n)}$ et $q=\frac{u}{\sqrt{(uu+n)}}$, erit

$$1+uu-vv=1-n, \quad v-qu=\frac{n}{\sqrt{(uu+n)}} \quad \text{et} \quad qq-1=\frac{-n}{uu+n}.$$

Quare habebitur

$$lx = la - l\sqrt{(1-n)} - \frac{1}{1-n}\int \frac{du\sqrt{(nn-n)}}{\sqrt{(uu+n)}} = lb + \frac{\sqrt{n}}{\sqrt{(n-1)}}\, l\left(u + \sqrt{(uu+n)}\right)$$

hincque

$$\frac{x}{b} = \left(\frac{y + \sqrt{(yy+nxx)}}{x}\right)^{\sqrt{\frac{n}{n-1}}}.$$

Quoties ergo $\frac{n}{n-1}$ est numerus quadratus, aequatio inter x et y prodit algebraica.

Sit $\sqrt{\frac{n}{n-1}} = m$; erit $n = \frac{mm}{mm-1}$ et $ss = yy + \frac{mmxx}{mm-1}$, cui conditioni satisfit hac aequatione algebraica

$$x^{m+1} = b\left(y + \sqrt{\left(yy + \frac{mmxx}{mm-1}\right)}\right)^m,$$

quae transformatur in

$$x^{\frac{2}{m}} - 2b^{\frac{1}{m}} x^{\frac{1-m}{m}} y = \frac{mm}{mm-1} b^{\frac{2}{m}} \quad \text{seu} \quad y = \frac{(mm-1)x^{\frac{2}{m}} - mmb^{\frac{2}{m}}}{2(mm-1)b^{\frac{1}{m}} x^{\frac{1-m}{m}}}.$$

COROLLARIUM

694. Ponamus $m = \frac{1}{n}$, ac si fuerit

$$y = \frac{b^{2n} + (nn-1)x^{2n}}{2(nn-1)b^n x^{n-1}},$$

erit

$$ss = yy - \frac{xx}{nn-1} \quad \text{seu} \quad s = \sqrt{\left(yy - \frac{xx}{nn-1}\right)}.$$

Quare si $y = \frac{b^4 + 3x^4}{6bbx}$, est $s = \sqrt{\left(yy - \frac{xx}{3}\right)}$.

PROBLEMA 91

695. *Si posito* $\frac{dy}{dx} = p$ *eiusmodi detur aequatio inter* x, y *et* p, *in qua altera variabilis* y *unicam tantum habeat dimensionem, invenire relationem inter binas variabiles* x *et* y.

SOLUTIO

Hinc ergo y aequabitur functioni cuipiam ipsarum x et p, unde differentiando fiet $dy = Pdx + Qdp$. Cum igitur sit $dy = pdx$, habebitur haec aequatio differentialis $(P-p)dx+Qdp=0$, quam integrari oportet. Quoniam tantum duas continet variabiles x et p et differentialia simpliciter involvit, eius resolutio per methodos supra expositas est tentanda.

Primo ergo resolutio succedet, si fuerit $P=p$ ideoque $dy=pdx+Qdp$. Quod evenit, si y per x et p ita determinetur, ut sit $y=px+\Pi$ denotante Π functionem quamcunque ipsius p. Tum ergo erit $Q=x+\frac{d\Pi}{dp}$, et cum solutio ab ista aequatione $Qdp=0$ pendeat, erit vel $dp=0$ hincque $p=\alpha$ seu $y=\alpha x+\beta$, ubi altera constantium α et β per ipsam aequationem propositam determinatur, dum posito $p=\alpha$ fit $\beta=\Pi$; vel erit $Q=0$ ideoque $x=-\frac{d\Pi}{dp}$ et $y=-\frac{pd\Pi}{dp}+\Pi$, ubi ergo utraque solutio est algebraica, si modo Π fuerit functio algebraica ipsius p.

Secundo aequatio $(P-p)dx+Qdp=0$ resolutionem admittet, si altera variabilis x cum suo differentiali dx unam dimensionem non superet. Evenit hoc, si fuerit $y=Px+\Pi$, dum P et Π sunt functiones ipsius p tantum; tum enim erit $P=P$ et $Q=\frac{xdP}{dp}+\frac{d\Pi}{dp}$ hincque haec habetur aequatio integranda

$$(P-p)dx+xdP+d\Pi=0 \quad \text{seu} \quad dx+\frac{xdP}{P-p}=-\frac{d\Pi}{P-p},$$

quae per $e^{\int\frac{dP}{P-p}}$ multiplicata dat

$$e^{\int\frac{dP}{P-p}}x=-\int e^{\int\frac{dP}{P-p}}\frac{d\Pi}{P-p}.$$

Sive ponatur $\frac{dP}{P-p}=\frac{dR}{R}$; erit aequatio integralis

$$Rx=C-\int\frac{Rd\Pi}{P-p}=C-\int\frac{d\Pi dR}{dP},$$

unde fit

$$x=\frac{C}{R}-\frac{1}{R}\int\frac{d\Pi dR}{dP} \quad \text{et} \quad y=\frac{CP}{R}+\Pi-\frac{P}{R}\int\frac{d\Pi dR}{dP}.$$

Tertio resolutio nullam habebit difficultatem, si denotantibus X et V functiones quascunque ipsius x fuerit $y=X+Vp$. Tum enim erit

$$dy=pdx=dX+Vdp+pdV$$

ideoque

$$dp+p\frac{dV-dx}{V}=-\frac{dX}{V};$$

sit $\frac{dx}{V}=\frac{dR}{R}$, ut R sit etiam functio ipsius x; erit $\frac{V}{R}p=C-\int\frac{dX}{R}$ seu

$$p=\frac{CR}{V}-\frac{R}{V}\int\frac{dX}{R} \quad \text{et} \quad y=X+CR-R\int\frac{dX}{R},$$

quae aequatio relationem inter x et y exprimit.

Quarto aequatio $(P-p)dx+Qdp=0$ resolutionem admittit, si fuerit homogenea. Cum ergo terminus pdx duas contineat dimensiones, hoc evenit, si totidem dimensiones et in reliquis terminis insint. Unde perspicuum est P et Q esse debere functiones homogeneas unius dimensionis ipsarum x et p. Quare si y ita per x et p definiatur, ut y aequetur functioni homogeneae duarum dimensionum ipsarum x et p, resolutio succedet. Quodsi enim fuerit $dy=Pdx+Qdp$, aequatio solutionem continens $(P-p)dx+Qdp=0$ erit homogenea fietque per se integrabilis, si dividatur per $(P-p)x+Qp$.

COROLLARIUM 1

696. Pro casu quarto si ponatur $y=zz$, aequatio proposita debet esse homogenea inter tres variabiles x, z et p. Unde si proponatur aequatio homogenea quaecunque inter x, z et p, in qua hae ternae litterae x, z et p ubique eundem dimensionum numerum constituant, problema semper resolutionem admittit.

COROLLARIUM 2

697. Simili modo conversis variabilibus si ponatur $x=vv$ et $\frac{dx}{dy}=q$, ut sit $p=\frac{1}{q}$, ac proponatur aequatio homogenea quaecunque inter y, v et q, problema itidem resolvi potest.

SCHOLION

698. Pro casu quarto, ut aequatio $(P-p)dx+Qdp=0$ fiat homogenea, conditiones magis amplificari possunt. Ponatur enim $x=v^\mu$ et $p=q^\nu$ sitque facta substitutione haec aequatio $\mu(P-q^\nu)v^{\mu-1}dv+\nu Qq^{\nu-1}dq=0$ homogenea inter v et q eritque P functio homogenea ν dimensionum et Q functio homogenea μ dimensionum. Cum iam sit

$$dy=Pdx+Qdp=\mu Pv^{\mu-1}dv+\nu Qq^{\nu-1}dq,$$

erit y functio homogenea $\mu+\nu$ dimensionum. Quare posito $y=z^{\mu+\nu}$ problema resolutionem admittit, si inter x, y et p eiusmodi relatio proponatur, ut positis $y=z^{\mu+\nu}$, $x=v^{\mu}$ et $p=q^{\nu}$ habeatur aequatio homogenea inter ternas quantitates z, v et q, ita ut dimensionum ab iis formatarum numerus ubique sit idem. Ac si proposita fuerit huiusmodi aequatio homogenea inter z, v et q, solutio problematis ita expedietur.

Cum sit $dy=pdx$, erit

$$(\mu+\nu)z^{\mu+\nu-1}dz=\mu v^{\mu-1}q^{\nu}dv;$$

ponatur iam $z=rq$ et $v=sq$ et aequatio proposita tantum duas litteras r et s continebit, ex qua alteram per alteram definire licet; tum autem per has substitutiones prodibit haec aequatio

$$(\mu+\nu)r^{\mu+\nu-1}q^{\mu+\nu-1}(rdq+qdr)=\mu s^{\mu-1}q^{\mu+\nu-1}(sdq+qds),$$

ex qua oritur

$$\frac{dq}{q}=\frac{\mu s^{\mu-1}ds-(\mu+\nu)r^{\mu+\nu-1}dr}{(\mu+\nu)r^{\mu+\nu}-\mu s^{\mu}},$$

quae est aequatio differentialis separata, quoniam s per r datur. Quin etiam bini casus allati manifesto continentur in formulis $y=z^{\mu+\nu}$, $x=v^{\mu}$ et $p=q^{\nu}$, prior scilicet, si $\mu=1$ et $\nu=1$, posterior vero, si $\mu=2$ et $\nu=-1$.

Hos igitur casus perinde ac praecedentes exemplis illustrari conveniet, quorum primus praecipue est memorabilis, cum per differentiationem aequationis propositae $y=px+\Pi$ statim praebeat aequationem integralem quaesitam neque integratione omnino sit opus, siquidem alteram solutionem ex $dp=0$ natam excludamus.

EXEMPLUM 1

699. *Proposita aequatione differentiali* $ydx-xdy=a\sqrt{(dx^2+dy^2)}$ *eius integrale invenire.*

Posito $\frac{dy}{dx}=p$ fit $y-px=a\sqrt{(1+pp)}$, quae aequatio differentiata ob $dy=pdx$ dat $-xdp=\frac{apdp}{\sqrt{(1+pp)}}$; quae cum sit divisibilis per dp, praebet primo $p=\alpha$ hincque

$$y=\alpha x+a\sqrt{(1+\alpha\alpha)}.$$

Alter vero factor suppeditat

$$x = \frac{-ap}{\sqrt{(1+pp)}} \quad \text{hincque} \quad y = \frac{-app}{\sqrt{(1+pp)}} + a\sqrt{(1+pp)} = \frac{a}{\sqrt{(1+pp)}},$$

unde fit

$$xx + yy = aa,$$

quae est etiam aequatio integralis; sed quia novam constantem non involvit, non pro completo integrali haberi potest. Integrale autem completum duas aequationes complectitur, scilicet

$$y = \alpha x + a\sqrt{(1 + \alpha\alpha)} \quad \text{et} \quad xx + yy = aa,$$

quae in hac una comprehendi possunt

$$((y - \alpha x)^2 - aa(1 + \alpha\alpha))(xx + yy - aa) = 0.$$

SCHOLION

700. Nisi hoc modo operatio instituatur, solutio huius quaestionis fit satis difficilis. Si enim aequationem differentialem $ydx - xdy = a\sqrt{(dx^2 + dy^2)}$ quadrando ab irrationalitate liberemus indeque rationem $\frac{dy}{dx}$ per radicis extractionem definiamus, fit

$$(xx - aa)dy - xydx = \pm\, adx\sqrt{(xx + yy - aa)},$$

quae aequatio per methodos cognitas difficulter tractatur. Multiplicator quidem inveniri potest utrumque membrum per se integrabile reddens; prius enim membrum $(xx - aa)dy - xydx$ divisum per $y(xx - aa)$ fit integrabile integrali existente $l\frac{y}{\sqrt{(xx - aa)}}$; unde in genere multiplicator id integrabile reddens est

$$\frac{1}{y(xx - aa)}\,\Phi : \frac{y}{\sqrt{(xx - aa)}},$$

quae functio ita determinari debet, ut eodem multiplicatore quoque alterum membrum $adx\sqrt{(xx + yy - aa)}$ fiat integrabile. Talis autem multiplicator est

$$\frac{1}{y(xx - aa)} \cdot \frac{y}{\sqrt{(xx + yy - aa)}} = \frac{1}{(xx - aa)\sqrt{(xx + yy - aa)}},$$

quo fit

$$\frac{(xx-aa)\,dy-xydx}{(xx-aa)\sqrt{(xx+yy-aa)}}=\frac{\pm adx}{xx-aa}.$$

Iam ad integrale prioris membri investigandum spectetur x ut constans eritque integrale

$$=l\left(y+\sqrt{(xx+yy-aa)}\right)+X$$

denotante X functionem quampiam ipsius x ita comparatam, ut sumta iam y constante fiat

$$\frac{xdx}{(y+\sqrt{(xx+yy-aa)})\sqrt{(xx+yy-aa)}}+dX=\frac{-xydx}{(xx-aa)\sqrt{(xx+yy-aa)}}$$

seu

$$\frac{-xdx(y-\sqrt{(xx+yy-aa)})}{(xx-aa)\sqrt{(xx+yy-aa)}}+dX=\frac{-xydx}{(xx-aa)\sqrt{(xx+yy-aa)}},$$

unde fit

$$dX=\frac{-xdx}{xx-aa}\quad \text{et}\quad X=l\frac{C}{\sqrt{(xx-aa)}}.$$

Quare integrale quaesitum est

$$l\left(y+\sqrt{(xx+yy-aa)}\right)+l\frac{C}{\sqrt{(xx-aa)}}=\pm\frac{1}{2}l\frac{x+a}{x-a}$$

seu

$$\frac{y+\sqrt{(xx+yy-aa)}}{\sqrt{(xx-aa)}}=\alpha\sqrt{\frac{x+a}{x-a}}\quad \text{vel}\quad =\alpha\sqrt{\frac{x-a}{x+a}},$$

unde fit

$$y+\sqrt{(xx+yy-aa)}=\alpha(x\pm a)$$

hincque

$$xx-aa=\alpha\alpha(x\pm a)^2-2\alpha(x\pm a)y$$

vel

$$x\mp a=\alpha\alpha(x\pm a)-2\alpha y;$$

quae autem tantum est altera binarum aequationum integralium, altera autem aequatio integralis $xx+yy=aa$ iam quasi per divisionem de calculo sublata est censenda.

Caeterum eadem solutio aequationis

$$(aa-xx)dy+xydx=\pm adx\sqrt{(xx+yy-aa)}$$

facilius instituitur ponendo $y = u\sqrt{(aa - xx)}$, unde fit

$$(aa - xx)^{\frac{3}{2}} du = \pm\, adx \sqrt{(aa - xx)(uu - 1)} \quad \text{seu} \quad \frac{du}{\sqrt{(uu-1)}} = \frac{\pm\, adx}{aa - xx},$$

cui quidem satisfit sumendo $u = 1$, neque tamen hic casus in aequatione integrali continetur, uti supra [§ 562] iam ostendimus. Ex quo suspicari liceret alteram solutionem $xx + yy = aa$ adeo esse excludendam, quod tamen secus se habere deprehenditur, si ipsam aequationem primariam $\frac{ydx - xdy}{\sqrt{(dx^2 + dy^2)}} = a$ perpendamus. Si enim x et y sint coordinatae rectangulae lineae curvae, formula $\frac{ydx - xdy}{\sqrt{(dx^2 + dy^2)}}$ exprimit perpendiculum ex origine coordinatarum in tangentem demissum, quod ergo constans esse debet. Hoc autem evenire in circulo origine in centro constituta, dum aequatio fit $xx + yy = aa$, per se est manifestum. Atque hinc realitas harum solutionum, quae minus congruae videri poterant, confirmatur, etiamsi earum ratio haud satis clare perspiciatur.

EXEMPLUM 2

701. *Proposita aequatione differentiali* $ydx - xdy = \frac{a(dx^2 + dy^2)}{dx}$ *eius integrale invenire.*

Posito $dy = pdx$ fit $y - px = a(1 + pp)$ et differentiando $-xdp = 2apdp$, unde concluditur vel $dp = 0$ et $p = \alpha$ hincque $y = \alpha x + a(1 + \alpha\alpha)$ vel $x = -2ap$ et $y = a(1 - pp)$ sicque ob $p = \frac{-x}{2a}$ habebitur $4ay = 4aa - xx$, quae aequatio ad geometriam translata illam conditionem omnino adimplet.

Ex aequatione autem proposita radicem extrahendo reperitur

$$2ady + xdx = dx\sqrt{(xx + 4ay - 4aa)},$$

quae posito $y = u(4aa - xx)$ abit in

$$2adu(4aa - xx) - xdx(4au - 1) = dx\sqrt{(4aa - xx)(4au - 1)}$$

haecque posito $4au - 1 = tt$ in

$$tdt(4aa - xx) - ttxdx = tdx\sqrt{(4aa - xx)};$$

quae cum sit divisibilis per t, concludere licet $t = 0$ ideoque $u = \frac{1}{4a}$ atque hinc $4ay = 4aa - xx$.

EXEMPLUM 3

702. *Proposita aequatione differentiali* $y\,dx - x\,dy = a\sqrt[3]{(dx^3 + dy^3)}$ *eius integrale assignare.*

Haec aequatio more consueto, si rationem $\frac{dy}{dx}$ inde extrahere vellemus, vix tractari posset. Posito autem $dy = p\,dx$ fit $y - px = a\sqrt[3]{(1 + p^3)}$ et differentiando

$$x\,dp = \frac{-\,app\,dp}{\sqrt[3]{(1+p^3)^2}},$$

unde duplex conclusio deducitur: vel $dp = 0$ et $p = \alpha$ sicque

$$y = \alpha x + a\sqrt[3]{(1 + \alpha^3)},$$

vel

$$x = \frac{-\,app}{\sqrt[3]{(1+p^3)^2}} \quad \text{et} \quad y = \frac{a}{\sqrt[3]{(1+p^3)^2}},$$

unde fit $pp = -\frac{x}{y}$, et ob $y^3(1 + p^3)^2 = a^3$ erit $p^3 = \frac{a\sqrt{a}}{y\sqrt{y}} - 1$ hincque

$$\frac{(a\sqrt{a} - y\sqrt{y})^2}{y^3} = -\frac{x^3}{y^3} \quad \text{seu} \quad x^3 + (a\sqrt{a} - y\sqrt{y})^2 = 0.$$

EXEMPLUM 4

703. *Proposita aequatione* $y\,dx - nx\,dy = a\sqrt{(dx^2 + dy^2)}$ *eius integrale invenire.*

Posito $dy = p\,dx$ habetur $y - npx = a\sqrt{(1 + pp)}$, unde differentiando elicitur

$$(1-n)p\,dx - nx\,dp = \frac{ap\,dp}{\sqrt{(1+pp)}} \quad \text{sive} \quad dx - \frac{nx\,dp}{(1-n)p} = \frac{a\,dp}{(1-n)\sqrt{(1+pp)}},$$

quae per $p^{\frac{n}{n-1}}$ multiplicata et integrata praebet

$$p^{\frac{n}{n-1}}x = \frac{a}{1-n}\int \frac{p^{\frac{n}{n-1}}dp}{\sqrt{(1+pp)}}.$$

Hinc deducimus casus sequentes integrationem admittentes[1]):

$$\text{si} \quad n = \frac{3}{2}, \quad p^3 x = C - \frac{2}{3} a \left(pp - \frac{2}{1}\right) \sqrt{(1+pp)},$$

$$\text{si} \quad n = \frac{5}{4}, \quad p^5 x = C - \frac{4}{5} a \left(p^4 - \frac{4}{3} p^2 + \frac{4 \cdot 2}{3 \cdot 1}\right) \sqrt{(1+pp)},$$

$$\text{si} \quad n = \frac{7}{6}, \quad p^7 x = C - \frac{6}{7} a \left(p^6 - \frac{6}{5} p^4 + \frac{6 \cdot 4}{5 \cdot 3} p^2 - \frac{6 \cdot 4 \cdot 2}{5 \cdot 3 \cdot 1}\right) \sqrt{(1+pp)},$$

ac si $n = \frac{2\lambda+1}{2\lambda}$, erit

$$y = px + a\sqrt{(1+pp)} + \frac{px}{2\lambda}$$

et

$$x = \frac{C}{p^{2\lambda+1}} - \frac{2\lambda a}{(2\lambda+1)p}\left(1 - \frac{2\lambda}{(2\lambda-1)pp} + \frac{2\lambda(2\lambda-2)}{(2\lambda-1)(2\lambda-3)p^4} - \text{etc.}\right)\sqrt{(1+pp)}.$$

Quodsi ergo sumatur $\lambda = \infty$, ut sit $n = 1$, erit

$$y = px + a\sqrt{(1+pp)} \quad \text{et} \quad x = \frac{C}{p^{2\lambda+1}} - \frac{ap}{\sqrt{(1+pp)}},$$

unde, si constans C sit $= 0$, statim sequitur solutio superior $xx + yy = aa$. At si constans C non evanescat, minimum discrimen in quantitate p infinitam varietatem ipsi x inducit. Quantumvis ergo x varietur, quantitas p ut constans spectari potest, unde posito $p = \alpha$ altera solutio $y = \alpha x + a\sqrt{(1 + \alpha\alpha)}$ obtinetur. Hinc ergo dubium supra circa Exemplum 1 natum non mediocriter illustratur.

EXEMPLUM 5

704. *Proposita aequatione differentiali* $A\,dy^n = (Bx^\alpha + Cy^\beta)dx^n$ *existente* $n = \frac{\alpha\beta}{\alpha-\beta}$ *eius integrale investigare.*

Posito $\frac{dy}{dx} = p$ erit $Ap^n = Bx^\alpha + Cy^\beta$. Ponamus iam $p = q^{\alpha\beta}$, $x = v^{\beta n}$ et $y = z^{\alpha n}$, ut habeamus hanc aequationem homogeneam $Aq^{\alpha\beta n} = Bv^{\alpha\beta n} + Cz^{\alpha\beta n}$, quae positis $z = rq$ et $v = sq$ abit in $A = Bs^{\alpha\beta n} + Cr^{\alpha\beta n}$. Cum vero sit

$$dy = \alpha n z^{\alpha n-1} dz = \alpha n r^{\alpha n-1} q^{\alpha n-1}(r\,dq + q\,dr)$$

et

$$p\,dx = \beta n v^{\beta n-1} q^{\alpha\beta} dv = \beta n s^{\beta n-1} q^{\alpha\beta+\beta n-1}(s\,dq + q\,ds),$$

1) Cf. formulam I § 118. F. E.

erit

$$\alpha r^{\alpha n-1}(rdq+qdr)=\beta s^{\beta n-1}q^{\alpha\beta+\beta n-\alpha n}(sdq+qds).$$

Est vero per hypothesin $\alpha\beta+\beta n-\alpha n=0$, unde oritur

$$\alpha r^{\alpha n}dq+\alpha r^{\alpha n-1}qdr=\beta s^{\beta n}dq+\beta s^{\beta n-1}qds$$

hincque

$$\frac{dq}{q}=\frac{\alpha r^{\alpha n-1}dr-\beta s^{\beta n-1}ds}{\beta s^{\beta n}-\alpha r^{\alpha n}}.$$

At est $s^{\beta n}=\left(\frac{A-Cr^{\alpha\beta n}}{B}\right)^{\frac{1}{\alpha}}$ hincque

$$\beta s^{\beta n-1}ds=-\frac{\beta C}{B}r^{\alpha\beta n-1}dr\left(\frac{A-Cr^{\alpha\beta n}}{B}\right)^{\frac{1-\alpha}{\alpha}},$$

unde fit

$$\frac{dq}{q}=\frac{\alpha r^{\alpha n-1}dr+\frac{\beta C}{B}r^{\alpha\beta n-1}dr\left(\frac{A-Cr^{\alpha\beta n}}{B}\right)^{\frac{1-\alpha}{\alpha}}}{\beta\left(\frac{A-Cr^{\alpha\beta n}}{B}\right)^{\frac{1}{\alpha}}-\alpha r^{\alpha n}}.$$

Facilius autem calculus hoc modo instituetur. Sumto $A=1$ erit

$$p=\frac{dy}{dx}=(Bx^{\alpha}+Cy^{\beta})^{\frac{1}{n}};$$

sit $y=x^{\frac{\alpha}{\beta}}u$; fiet

$$x^{\frac{\alpha}{\beta}}du+\frac{\alpha}{\beta}x^{\frac{\alpha-\beta}{\beta}}udx=x^{\frac{\alpha}{n}}dx(B+Cu^{\beta})^{\frac{1}{n}},$$

quae aequatio, cum sit $\frac{\alpha}{n}=\frac{\alpha-\beta}{\beta}$, abit in hanc

$$\beta xdu+\alpha udx=\beta dx(B+Cu^{\beta})^{\frac{1}{n}},$$

unde fit

$$\frac{dx}{x}=\frac{\beta du}{\beta(B+Cu^{\beta})^{\frac{1}{n}}-\alpha u},$$

sicque x per u determinatur, et quia $u=x^{-\frac{\alpha}{\beta}}y$, habebitur aequatio inter x et y.

SCHOLION

705. Hoc igitur modo operationem institui conveniet, quando inter binas variabiles x et y una cum differentialium ratione $\frac{dy}{dx} = p$ eiusmodi relatio proponitur, ex qua valor ipsius p commode elici non potest. Tum ergo calculum ita tractari oportet, ut per differentiationem ponendo $dy = pdx$ vel $dx = \frac{dy}{p}$ tandem perveniatur ad aequationem differentialem simplicem inter duas tantum variabiles, quem in finem etiam saepe idoneis substitutionibus uti necesse est.

Atque hucusque fere Geometris in resolutione aequationum differentialium primi gradus etiamnum pertingere licuit; vix enim ulla via integralia investigandi adhuc quidem adhibita hic praetermissa videtur. Num autem multo maiorem calculi integralis promotionem sperare liceat, vix equidem affirmaverim, cum plurima extent inventa, quae ante vires ingenii humani superare videbantur.

Cum igitur calculum integralem in duos libros sim partitus, quorum prior circa relationem binarum tantum variabilium, posterior vero ternarum pluriumve versatur, atque iam libri primi partem priorem in differentialibus primi ordinis constitutam hic pro viribus exposuerim, ad eius alteram partem progredior, in qua binarum variabilium relatio ex data differentialium secundi altiorisve ordinis conditione requiritur.

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